Observer Design for Unknown Input Nonlinear Descriptor Systems via Convex Optimization

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***Abstract*—This paper treats the design problem of full-order observers for nonlinear descriptor systems with unknown input (UI). Depending on the available knowledge on the UI dynamics, two cases are considered. First, a UI proportional observer (UIPO) is proposed when the spectral domain of the UI is unknown. Second, a PIO is proposed when the spectral domain of the UI is in the low frequency range. Sufficient conditions for the existence and stability of such observers are given and proved. Based on the linear matrix inequality (LMI) approach, an algorithm is presented to compute the observer gain matrix that achieves the asymptotic stability objective. An example is included to illustrate the method.**

***Index Terms*—Linear matrix inequalities (LMIs), Lipschitz nonlinear descriptor systems, proportional integral observers, unknown input observers.**

I. INTRODUCTION

Observer design for linear systems has received great attention in the literature and some extensions have been proposed for the cases of unknown inputs [8] and descriptor systems [9], [14]. For physical processes that are described by nonlinear models, three approaches can be distinguished for the design of nonlinear observers. The first one is based on a nonlinear transformation using Lie algebra to bring the system into a canonical form and then use linear techniques to design the state observer. Necessary and sufficient conditions for a nonlinear system to be equivalent to the canonical form have been established in [12] and [13] but this approach necessitates conservative conditions. The second approach is based on the linearized model. In spite of the local convergence of this method, it is widely used in practice and generally gives better results under less restrictive conditions than the first approach. In [25], the authors have established a necessary condition for the existence of a local exponential observer for nonlinear systems. The third approach treats the observer design problem for a class of nonlinear systems which are composed of a linear part and a vector of nonlinear functions. It was developed by [1], [7], [17], [24] where sufficient conditions for global stability of the observer were established.

However, few works have been done to extend the methods mentioned above to the general representation of nonlinear descriptor systems. In [10] and [5], a linearization is used to design state observers for nonlinear descriptor systems in ac/dc converter applications without unknown inputs (UIs).

The work presented here considers a general class of nonlinear descriptor systems subject to UIs and unknown measurement disturbances where nonlinearities are assumed to be Lipschitz. Before presenting the main results, a brief review of the PIO is presented. PIO are used to attenuate the effect of UI, nonlinearities and uncertain parameters. PIO have been applied in many applications such as robust controller design [3], fault diagnosis [15], loop transfer recovery design [16], parameter estimation [21], and state and fault estimation [11].

In this paper, two rigorous estimation algorithms that are robust to both process and sensor noise are proposed for a class of UI nonlinear descriptor systems. The first one consists in designing a UI observer which gives a perfect UI decoupled state estimation, while the second one consists in designing a PIO which attenuates the impact of disturbances in the low and high spectral domains.

Notations: $(.)^T$ is the transpose matrix. $(.) > 0$ denotes symmetric positive definite matrices. σ denotes singular values with $\underline{\sigma}$ the smallest and $\bar{\sigma}$ the largest singular values. $(.)^+$ is the generalized inverse matrix.

II. PROBLEM FORMULATION

Consider the nonlinear system of the form

$$\begin{aligned} E\dot{x} &= Ax + Fw + H\phi(x,u,t) \\ y &= Cx + Gw \end{aligned} \tag{1}$$

where E may be rank deficient, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^{n_u}$, $w \in \mathbb{R}^{n_w}$, $\phi : R^n \times R^{n_u} \times R \to R^n$ and $y \in \mathbb{R}^m$ denote, respectively, the state, the known input, the UI, the nonlinearity and the output vectors. E, A, $H \in R^{n\times n}$, $F \in R^{n\times n_w}$, $G \in R^{m\times n_w}$, and $C \in R^{m\times n}$ are known constant matrices. Before giving the main results, let us make the following well-known assumptions.

- A1) The nonlinearity $\phi(x,u,t)$ is globally Lipschitz in x with Lipschitz constant γ, i.e.,

$$\|\phi(x,u,t) - \phi(\hat{x},u,t)\| \leq \gamma\|x-\hat{x}\|, \forall u \in \mathbb{R}^{n_u}, t \in \mathbb{R}.$$

- A2)

$$\text{rank}\begin{bmatrix} F \\ G \end{bmatrix} = n_w \text{ and } \text{rank}[C \quad G] = m.$$

- A3a)

$$\text{rank}\begin{bmatrix} E & F & 0 \\ 0 & G & 0 \\ C & 0 & G \end{bmatrix} = n + \text{rank}\begin{bmatrix} F \\ G \end{bmatrix} + \text{rank}G.$$

- A3b)

$$\text{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n.$$

- A4a)

$$\text{rank}\begin{bmatrix} pE-A & -F \\ C & G \end{bmatrix} = n + \text{rank}\begin{bmatrix} F \\ G \end{bmatrix} \ \forall \mathbb{R}(p) \geq 0.$$

- A4b)

$$\text{rank}\begin{bmatrix} pE-A & -F \\ 0 & pI_{n_w} \\ C & G \end{bmatrix} = n + \text{rank}\begin{bmatrix} F \\ G \end{bmatrix} \ \forall \mathbb{R}(p) \geq 0.$$

Manuscript received April 30, 2004; revised January 27, 2005, June 16, 2005, and November 4, 2005. Recommended by Associate Editor S. Tarbouriech.

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Digital Object Identifier 10.1109/TAC.2006.876807

Like in [2], the measurement y is time integrated (i.e., $y_I = \int_0^t y dv \in \mathbb{R}^m$) in order to attenuate the noise impact in the estimation error (see the discussions in [2] and [21]). Therefore, (1) is transformed to the restricted system equivalence (r.s.e.)

$$\bar{E}\dot{\bar{x}} = \bar{A}\bar{x} + \bar{F}w + \bar{H}\phi(x,u,t)$$
$$y_I = C_I\bar{x} \quad y = \bar{C}\bar{x} + Gw \quad \breve{y} = \breve{C}\bar{x} + \breve{G}w$$

where

$C_I = [0_{m\times n}\ I_m]$, $\bar{C} = [C\ 0_{m\times m}]$, $\breve{y} = \begin{bmatrix} y_I \\ y \end{bmatrix}$, $\bar{x} = \begin{bmatrix} x \\ y_I \end{bmatrix} \in \mathbb{R}^{n+m}$, $\bar{F} = \begin{bmatrix} F \\ G \end{bmatrix}$, $\bar{E} = \begin{bmatrix} E & 0_{n\times m} \\ 0_{m\times n} & I_m \end{bmatrix}$, $\bar{A} = \begin{bmatrix} A & 0_{n\times m} \\ C & 0_{m\times m} \end{bmatrix}$, $\breve{G} = \begin{bmatrix} 0_{m\times n_w} \\ G \end{bmatrix}$, $\breve{C} = \begin{bmatrix} C_I \\ \bar{C} \end{bmatrix}$,

and

$$\bar{H} = \begin{bmatrix} H \\ 0_{m\times n} \end{bmatrix} \tag{2}$$

Objectives:

1) If any knowledge about the spectral domain of the UI w is given, then under A1, A2, A3a, and A4a the following UIPO is proposed:

$$\dot{z} = \pi z + K_{p_1}y_I + K_{p_2}\breve{y} + T\bar{H}\phi(\hat{x},u,t) \tag{3}$$
$$\hat{\bar{x}} = z + N\breve{y}, \quad \hat{x} = [I_n \quad 0]\hat{\bar{x}}$$

where π, K_{p_1}, K_{p_2}, T, and N are determined such that $\hat{x}$ asymptotically converges to x for any w and any initial condition (eventually in a given set if it consists of local convergence).

2) If the spectral domain of the UI w is in the low frequency range, then under A1, A2, A3b, and A4b the following PIO is proposed:

$$\dot{z} = \pi z + K_{p_1}y_I + K_{p_2}\breve{y} + T\bar{F}\hat{w} + T\bar{H}\phi(\hat{x},u,t)$$
$$\dot{\hat{w}} = K_I(y_I - C_I\hat{\bar{x}}) \tag{4}$$
$$\hat{\bar{x}} = z + N\breve{y}, \quad \hat{x} = [I_n \quad 0]\hat{\bar{x}}$$

where z, $\bar{x}$, $\hat{\bar{x}} \in \mathbb{R}^{n+m}$, $\hat{x} \in \mathbb{R}^n$, $\hat{w} \in \mathbb{R}^{n_w}$, and π, K_{p_1}, K_{p_2}, K_I, T, and N are unknown matrices which must be determined such that $\hat{\bar{x}}$, $\hat{x}$ and $\hat{w}$ asymptotically converge to $\bar{x}$, x and w respectively for any initial condition (eventually in a given set if it consists of local convergence).

3) Find the largest Lipschitz constant γ_1 in the nonlinearity for which the observer (3) or (4) exists for system (2) which is an r.s.e. of (1).

Remark 1: Consider the nonlinear system $E\dot{x} = Ax + f(x,t)$ where $x = (x_1\ x_2)^T$ and $f(x,t) = (tx_1(t)\ 0.3\sin x_2(t))^T$. It is clear that $f(x,t)$ is not fully Lipschitz due to presence of the term $tx_1(t)$. However, $E\dot{x} = Ax + f(x,t)$ can be expressed as (1), where $F = [1\ 0]^T$, $w(t) = tx_1(t)$, $H = [0\ 1]^T$, $\phi(x,u,t) = 0.3\sin x_2(t)$ and where $\gamma = 0.3$ is the Lipschitz constant. Thus, it is clear that the class of nonlinear systems considered in this paper is more general than [7], [17], and [24].

Remark 2: Consider the general nonlinear system

$$E\dot{x} = f(x) + g(x)u + Fw \tag{5}$$
$$y = Cx + Gw$$

where $f(.)$ and $g(.)$ are continuously differentiable function with $f(0) = 0$. Let us denote $A = (\partial f/\partial x)|_{x=0}$, $B = g(0)$. Then, the given system (5) can be expanded as (1) where $\phi(x,u,t) = Bu + f_1(x) + g_1(x)u$, $H = I_n$, and where $f_1(x)$ (respectively, $g_1(x)$) is obtained from expanding $f(x)$ (respectively, $g_1(x)$) in a Taylor series around $x = 0$.

Before giving the main results, we introduce the following notations for UIPO and PIO:

$$\tilde{\phi} = \phi(x,u,t) - \phi(\hat{x},u,t), \alpha_1 = \Psi_1\Theta_1^+\varphi_1, \alpha = \underline{\sigma}(-\chi_1)$$
$$e_{\bar{x}} = \bar{x} - \hat{\bar{x}} \in \mathbb{R}^{n+m}, \alpha_2 = \Psi_1\Theta_1^+\varphi_2, U_1 = P_1 Z_1$$
$$\Psi_1 = \left[I_{n+m} \quad 0_{(n+m)\times(n+m+2n_w)}\right], \underline{\beta} = \underline{\sigma}(P_1)$$
$$\chi_1 = \alpha_1^T P_1 - \beta_1^T U_1^T + P_1\alpha_1 - U_1\beta_1 + I_{n+m} + \bar{\chi}_1,$$
$$\bar{\chi}_1 = \gamma^2(P_1\alpha_2 - U_1\beta_2)\left(\alpha_2^T P_1 - \beta_2^T U_1^T\right), \bar{\beta} = \bar{\sigma}(P_1),$$
$$\beta_1 = \left(I_{2n+5m} - \Theta_1\Theta_1^+\right), \varphi_1\beta_2 = \left(I_{2n+5m} - \Theta_1\Theta_1^+\right)\varphi_2,$$
$$\varphi_2^T = \left[\bar{H}^T \quad 0_{2m\times n}^T \quad 0_{m\times n}^T \quad 0_{(n+m)\times n}^T\right],$$
$$\Theta_1 = \begin{bmatrix} \bar{E} & \bar{A} & \bar{F} & 0 \\ \breve{C} & 0 & 0 & \breve{G} \\ 0 & -C_I & 0 & 0 \\ 0 & -I_{n+m} & 0 & 0 \end{bmatrix} \varphi_1 = \begin{bmatrix} \bar{A} \\ 0_{2m\times(n+m)} \\ -C_I \\ 0_{(n+m)\times(n+m)} \end{bmatrix},$$

and

$$A_e = \begin{bmatrix} T\bar{A} & T\bar{F} \\ 0_{n_w\times(n+m)} & 0_{n_w\times n_w} \end{bmatrix}, \Theta_2 = \begin{bmatrix} \bar{E} \\ \breve{C} \end{bmatrix},$$
$$T_e = \begin{bmatrix} T\bar{H} \\ 0_{n_w\times n} \end{bmatrix}, K_e = \begin{bmatrix} K_{p1} \\ K_I \end{bmatrix}, \Psi_2 = I_{n+m},$$
$$C_e = [C_I \quad 0_{m\times n_w}], U_2 = P_2K_e e_w = w - \hat{w},$$
$$e^T = \left[e_{\bar{x}}^T | e_w^T\right], \bar{\chi}_2 = P_2T_eT_e^T P_2 + \gamma^2 I_{n+m+n_w},$$
$$\chi_2 = A_e^T P_2 + P_2A_e - U_2C_e - C_e^T U_2^T + \bar{\chi}_2.$$

III. Observers Design

In this section, a new method is presented to design both UIPO and PIO for (1).

A. UIPO

Theorem 1: If

1) there exists matrices T, N, K_{p_1}, π such that

$$T\bar{E} + N\breve{C} = I_{n+m} \tag{6}$$
$$\pi = T\bar{A} - K_{p_1}C_I \tag{7}$$
$$K_{p_2} = \pi N \tag{8}$$
$$N\breve{G} = 0 \tag{9}$$
$$T\bar{F} = 0; \tag{10}$$

2) there exists a positive–definite matrix P_1, and a matrix $U_1 \in \mathbb{R}^{(n+m)\times(2n+5m)}$ such that the optimization problem

$$\min_{P_1,U_1} \quad \sigma \tag{11}$$

subject to

$$\begin{pmatrix} P_1\alpha_1 - U_1\beta_1 + \alpha_1^T P_1 - \beta_1^T U_1^T + I & P_1\alpha_2 - U_1\beta_2 \\ * & -\sigma I_n \end{pmatrix} < 0 \tag{12}$$

is feasible, then objectives 1) and 3) hold and the UIPO (3) is a global observer (i.e., asymptotically estimates x for any w and any initial estimate error) if $\gamma \le \gamma_1 = \sigma^{-1/2}$. Moreover the resulting observer gain $Z_1 = P_1^{-1}U_1$ ensures that the estimation error is exponentially stable, i.e.,

$$\|e_{\bar{x}}\| \le \sqrt{\underline{\beta}^{-1}V\left(e_{\bar{x}}(0)\right)}\exp^{-\frac{1}{2}\alpha\bar{\beta}^{-1}t}. \tag{13}$$

One can note that if $\gamma > \gamma_1$, one can choose, based on the obtained minimum value σ, a diagonal coordinate transformation [1], [18] to reduce the Lipschitz constant $\gamma(\leq \sigma^{-1/2})$.

Remark 3: The assumption that ϕ is Lipschitz globally (see the definition in [18]) may be relaxed to assume that ϕ is only locally Lipschitz. All the results will then be valid in some local neighborhood around a nominal point. In that case, the proposed observer produces local convergence of the observer error, the region of stability can be computed and its computation is shown in the last section of [7].

Proof—Part 1): Suppose that (6) and (9) hold, then the state estimation error $e_{\bar{x}}$ becomes $e_{\bar{x}} = T\bar{E}\bar{x} - z$. In this case, the dynamics of the estimation error $e_{\bar{x}}$ is described by

$$\dot{e}_{\bar{x}} = \pi e_{\bar{x}} + (T\bar{F} - K_{p_2}\breve{G})w + T\bar{H}\tilde{\phi} + (T\bar{A} - \pi T\bar{E} - K_{p_2}\breve{C} - K_{p_1}C_I)\bar{x}.$$

It follows from (6)–(10) that

$$\dot{e}_{\bar{x}} = (T\bar{A} - K_{p1}C_I)e_{\bar{x}} + T\bar{H}\tilde{\phi}. \quad (14)$$

Rewriting (14) and (6), (7), (9), (10), respectively, as

$$\dot{e}_{\bar{x}} = [T \quad N \quad K_{p1} \quad \pi]\varphi_1 e_{\bar{x}} + [T \quad N \quad K_{p_1} \quad \pi]\varphi_2\tilde{\phi} \quad (15)$$

$$[T \quad N \quad K_{p_1} \quad \pi]\Theta_1 = \Psi_1. \quad (16)$$

The solution of (16) depends on the rank of matrix Θ_1. A solution exists if and only if (iff) [19]

$$\operatorname{rank}\begin{bmatrix}\Theta_1\\ \Psi_1\end{bmatrix} = \operatorname{rank}\Theta_1. \quad (17)$$

Using relation (17) and the definition of matrices Θ_1 and Ψ_1, the necessary and sufficient conditions for the existence of a solution to (6), (7), (9), (10) of Theorem 1, or equivalently to the matrix equation (16) is A3a). Therefore, under assumption A3a), the general solution of (16) is

$$[T \quad N \quad K_{p_1} \quad \pi] = \Psi_1\Theta_1^+ - Z_1\left(I_{2n+5m} - \Theta_1\Theta_1^+\right) \quad (18)$$

where Z_1 is an arbitrary matrix of appropriate dimension. Substituting (18) into (15) gives

$$\dot{e}_{\bar{x}} = (\alpha_1 - Z_1\beta_1)e_{\bar{x}} + (\alpha_2 - Z_1\beta_2)\tilde{\phi}. \quad (19)$$

Proof—Part 2): Consider the quadratic Lyapunov function candidate $V(e_{\bar{x}}) = e_{\bar{x}}^T P_1 e_{\bar{x}}$ with $P_1 > 0$. The time derivative of $V(e_{\bar{x}})$ along system trajectories of (19) is

$$\dot{V}(e_{\bar{x}}) = e_{\bar{x}}^T\left(\alpha_1^T P_1 - \beta_1^T U_1^T + P_1\alpha_1 - U_1\beta_1\right)e_{\bar{x}} + 2e_{\bar{x}}^T(P_1\alpha_2 - U_1\beta_2)\tilde{\phi}.$$

From assumption A_1, we have

$$\begin{aligned}2e_{\bar{x}}^T(P_1\alpha_2 - U_1\beta_2)\tilde{\phi} &\leq 2\|\tilde{\phi}\|\left\|\left(\alpha_2^T P_1 - \beta_2^T U_1^T\right)e_{\bar{x}}\right\|\\ &\leq 2\gamma\|e_{\bar{x}}\|\left\|\left(\alpha_2^T P_1 - \beta_2^T U_1^T\right)e_{\bar{x}}\right\|\\ &\leq e_{\bar{x}}^T\bar{\chi}_1 e_{\bar{x}} + e_{\bar{x}}^T e_{\bar{x}}\end{aligned}$$

and thus $\dot{V}(e_{\bar{x}}) \leq e_{\bar{x}}^T\chi_1 e_{\bar{x}}$. The time derivative of $V(e_{\bar{x}})$ is negative definite for all $e_{\bar{x}} \neq 0$ if $\chi_1 < 0$. Using the Schur complement, $\chi_1 < 0$ if there exists a solution (P_1, U_1) to the optimization problem defined in Theorem 1. In addition, since $V(e_{\bar{x}}) \leq \bar{\beta}\|e_{\bar{x}}\|^2$ and $-\dot{V}(e_{\bar{x}}) \geq e_{\bar{x}}^T(-\chi_1)e_{\bar{x}} \geq \alpha\|e_{\bar{x}}\|^2$ then $\|e_{\bar{x}}\|^2 \geq \bar{\beta}^{-1}V(e_{\bar{x}})$ and $-\dot{V}(e_{\bar{x}}) \geq \alpha\bar{\beta}^{-1}V(e_{\bar{x}})$ which implies $V(e_{\bar{x}}(t)) < \exp^{-\alpha\bar{\beta}^{-1}t} \times V(e_{\bar{x}}(0))$. Finally, since $\underline{\beta}\|e_{\bar{x}}\|^2 \leq V(e_{\bar{x}})$, we deduce (13).

Remark 4: The existence of a solution $(P_1 > 0, U_1)$ of the LMI (12) requires that the matrix $\alpha_1 - Z_1\beta_1$ is Hurwitz since the first element in (12) implies $P_1(\alpha_1 - Z_1\beta_1) + (\alpha_1 - Z_1\beta_1)^T P_1 < 0$. Let us recall that $\alpha_1 - Z_1\beta_1$ can be stabilisable iff the pair $(\alpha_1 \ \beta_1)$ is detectable.

Now, we can establish the necessary conditions for the existence of the proposed observer (3).

Lemma 1: The necessary conditions for the existence of the observer (3) for system (1) are as follows.

1) A4a) which is equivalent to the detectability of the pair $(\alpha_1 \ \beta_1)$, i.e.

$$\operatorname{rank}\begin{bmatrix}pI_{n+m} - \alpha_1\\ \beta_1\end{bmatrix} = n + m, \quad \forall\,\mathbb{R}(p) \geq 0 \quad (20)$$

2) A3a)

Proof: The *part 1)* is done in the appendix while the proof of *part 2)* is done previously [see (17)].

The following algorithm summarizes the design procedure of the UIPO (3) for system (1).

Algorithm 1: Assume that lemma 1 is satisfied. Solve the convex optimization problem defined in theorem 1 and deduce $Z_1 = P_1^{-1}U_1$. Matrices T, N, K_{p_1}, π and K_{p_2} are computed from (18) and (8), respectively. □

B. PIO

If the spectral domain of the UI w is in the low frequency range, a general approach is possible by assuming that the disturbance is piecewise constant. See the remarks on PIO design in [22, Sec. 3.2] and [11, Rem. 2].

Theorem 2: Under $\dot{w} = 0$, if

1) there exists matrices T, N, K_{p_1}, π such that (6)–(8) hold
2) there exists a positive–definite matrix P_2, and a matrix $U_2 \in \mathbb{R}^{(n+m+n_w)\times m}$ such that the optimization problem

$$\min_{P_2,U_2} \quad \sigma$$

subject to

$$\begin{pmatrix}P_2A_e - U_2C_e + A_e^TP_2 - C_e^TU_2^T + I & P_2T_e\\ * & -\sigma I_n\end{pmatrix} < 0 \quad (21)$$

is feasible then objectives 2) and 3) hold and the PIO (4) is a global observer (i.e., asymptotically estimates x and w for any initial estimate error) if $\gamma \leq \gamma_1 = \sigma^{-1/2}$. Moreover, as in the previous section, the resulting observer gain $K_e = P_2^{-1}U_2$ ensures that the estimation error e (i.e., $e_{\bar{x}}$ and e_w) is exponentially stable.

Proof—Part 1): Suppose that (6) holds, then the state estimation error $e_{\bar{x}}$ becomes $e_{\bar{x}} = T\bar{E}\bar{x} - z - N\breve{G}w$. The dynamics of the estimation errors $e_{\bar{x}}$ and e_w become, respectively

$$\dot{e}_{\bar{x}} = \pi e_{\bar{x}} + T\bar{H}\tilde{\phi} + (T\bar{A} - \pi T\bar{E} - K_{p_1}C_I - K_{p_2}\breve{C})\bar{x} + (T\bar{F} + \pi N\breve{G} - K_{p_2}\breve{G})w - T\bar{F}\hat{w} \quad (22)$$

$$\dot{e}_w = -K_IC_Ie_{\bar{x}} \quad (23)$$

since $\dot{w} = 0$. It follows from (6)–(8) that

$$\dot{e}_{\bar{x}} = (T\bar{A} - K_{p1}C_I)e_{\bar{x}} + T\bar{F}e_w + T\bar{H}\tilde{\phi}. \quad (24)$$

Rewriting (6) as

$$[T \quad N]\,\Theta_2 = \Psi_2. \quad (25)$$

The solution of (25) depends on the rank of matrix Θ_2. A solution exists iff [19]

$$\text{rank}\begin{bmatrix}\Theta_2\\ \Psi_2\end{bmatrix} = \text{rank}\Theta_2 \quad (26)$$

which is obviously equivalent to the assumption A3b). Then, under A3b), the general solution of (26) is

$$[T \quad N] = \Psi_2\Theta_2^+ + Z_2\left(I_{n+3m} - \Theta_2\Theta_2^+\right) \quad (27)$$

where Z_2 is an arbitrary matrix fixed by the designer such that the matrix T is of maximal rank (i.e., $n+m$, see the discussion in [18]). Using the definition of A_e, K_e, C_e, T_e, and e, the relations (24) and (23) become

$$\dot{e} = (A_e - K_eC_e)e + T_e\tilde{\phi}. \quad (28)$$

Proof—Part 2): Consider the quadratic Lyapunov function candidate $V(e) = e^TP_2e$ with $P_2 > 0$. From assumption A_1, the time derivative of $V(e)$ along systems trajectories (28) gives $\dot{V}(e) < e^T\chi_2 e$. Using the Schur complement formula, $\dot{V}(e) < 0$ for all $e \neq 0$ if there exists a solution (P_2, U_2) to the optimization problem defined in Theorem 2.

Remark 5: The existence of a solution $(P_2 > 0, U_2)$ of the LMI (21) requires that the matrix $A_e - K_eC_e$ is Hurwitz since the first element in (21) implies $P_2(A_e - K_eC_e) + (A_e - K_eC_e)^TP_2 < 0$.

Now, we can establish the necessary conditions for the existence of the proposed observer (4).

Lemma 2: The necessary conditions for the existence of the observer (4) for system (1) are:

1) A4b), which under $rankT = m + n$ is equivalent to the detectability of the $pair(A_e, C_e)$, i.e.,

$$\text{rank}\begin{bmatrix}pI_{n+m+n_w} - A_e\\ C_e\end{bmatrix} = n+m+n_w, \quad \forall\,\mathbb{R}(p) \geq 0. \quad (29)$$

2) A3b).

Proof: The *part 1)* is done in the appendix while the proof of *part 2)* is done previously [see (26)].

Remark 6: The assumption A3b) is the same as the assumption b) given in [5]. A3b) can be relaxed to the impulse observability condition (see [11, Def. 3]) if rank$[E\ H] = rankE$ (see [11, Rem. 1 and Prop. 1] and [15], respectively) or if the nonlinear algebraic constraints obtained after the transformation $\breve{P}^T = [\breve{P}_1^T\ \ \breve{P}_2^T]$ can be rewritten with the known inputs [15], i.e. $\breve{P}_2H\phi(x,u,t) = f(y,u)$. Obviously, A3b) is less restrictive than A3a).

The following algorithm summarizes the design procedure of the PIO (4) for system (1).

Algorithm 2: Assume that Lemma 2 is satisfied. From (27), set Z_2 such that the matrix T is of maximal rank (i.e., $n+m$) and deduce T and N. Solve the convex optimization problem defined in Theorem 2 and deduce $\begin{bmatrix}K_{p1}\\ K_I\end{bmatrix} = K_e$. Matrices π and K_{p2} are deduced from (7) and (8), respectively. □

IV. Discussion

Since E is singular, system (1) can be rewritten as

$$\begin{cases} E_1\dot{x}_1 = A_1x_1 + F_1w + \bar{H}\phi(x,u,t)\\ y = C_1x_1\end{cases} \quad (30)$$

$$\begin{cases} E_2\dot{x}_2 = A_2x_2 + H\phi(x,u,t)\\ y = C_2x_2\end{cases} \quad (31)$$

where $x_1 = \begin{bmatrix}x\\ \varsigma\end{bmatrix} \in \mathbb{R}^{n+m}$, $x_2 = \begin{bmatrix}x\\ w\end{bmatrix} \in \mathbb{R}^{n+n_w}$, $A_1 = \begin{bmatrix}A & 0_{n\times m}\\ 0_{m\times n} & -I_m\end{bmatrix}$, $E_1 = \begin{bmatrix}E & 0_{n\times m}\\ 0_{m\times n} & 0_{m\times m}\end{bmatrix}$, $F_1 = \begin{bmatrix}F\\ G\end{bmatrix}$, $E_2 = [E\ \ 0_{n\times n_w}]$, $C_1 = [C\ I_m]$, $A_2 = [A\ F]$, and $C_2 = [C\ G]$.

1) Let $\Theta_1 = \begin{bmatrix}E_1 & F_1\\ C_1 & 0_{m\times n_w}\end{bmatrix}$ and $\Psi_1 = [I_{n+m}\ 0]$. An UIPO can be designed for (30) which satisfies the constraint $[T\ N]\Theta_1 = \Psi_1$ iff (17) holds. From Θ_1 and Ψ_1, the equality (17) is equivalent to rank $\begin{bmatrix}E & F\\ 0 & G\end{bmatrix} = n +$ rank $\begin{bmatrix}F\\ G\end{bmatrix}$. This implies that $rankE = n$ (i.e. not a descriptor system) which is more restrictive than $A3a$.
2) An UIPO can be designed for (31) iff rank$\begin{bmatrix}E_2\\ C_2\end{bmatrix} = n + n_w$ or, equivalently, iff rank $\begin{bmatrix}E & 0\\ C & G\end{bmatrix} = n + n_w$, which is also more restrictive than $A3a$.

V. Numerical Example

The following example illustrates respectively the UIPO (3) and PIO (4) estimation performance. Consider (1) with

$$A = \begin{bmatrix}-1 & 1 & 0 & 0\\ -1 & 0 & 0 & 1\\ 0 & -1 & -1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\ E = \begin{bmatrix}I_3 & 0\\ 0 & 0\end{bmatrix}$$

$$F = \begin{bmatrix}0 & 0\\ 0 & 1\\ 0 & 0\\ 0 & 1\end{bmatrix}\ \phi = \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\\ 1 & 0\end{bmatrix}u(t) + \begin{pmatrix}0\\ 0\\ 0\\ -\gamma\sin(x_3)\end{pmatrix}$$

$$C = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 1\end{bmatrix}\ G = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

where the Lipschitz constant is γ, fixed to 0.15 ($< \gamma_1$). In order to illustrate the robustness of each observer with respect to the noise and UI, we disturb the process by $w = (w_1, w_2)^T$, $bu = (bu_1, bu_2)^T$ and $by = (by_1, by_2)^T$ which represent respectively the UI and actuator and sensor noises. The components of the actuator and sensor noises are described in Fig. 1(b). The known input is $u = (u_1, u_2)^T$ where $u_1 = 0.7\sin 0.5t + bu_1$ and $u_2 = \sin 0.2t + bu_2$.

A. UIPO

Lemma 1 being satisfied for all p, we can arbitrary set the eigenvalues of $(\alpha_1 - Z_1\beta_1)$. For a good estimation performance, we propose to set the eigenvalues of the observer in a specified LMI region D [6]. Matrix $(\alpha_1 - Z_1\beta_1)$ has all its eigenvalues in the vertical strip defined by

$$D = \{x + jy \in C : -h_1 < x < -h_2\}, h_1, h_2 \in \Re \quad (32)$$

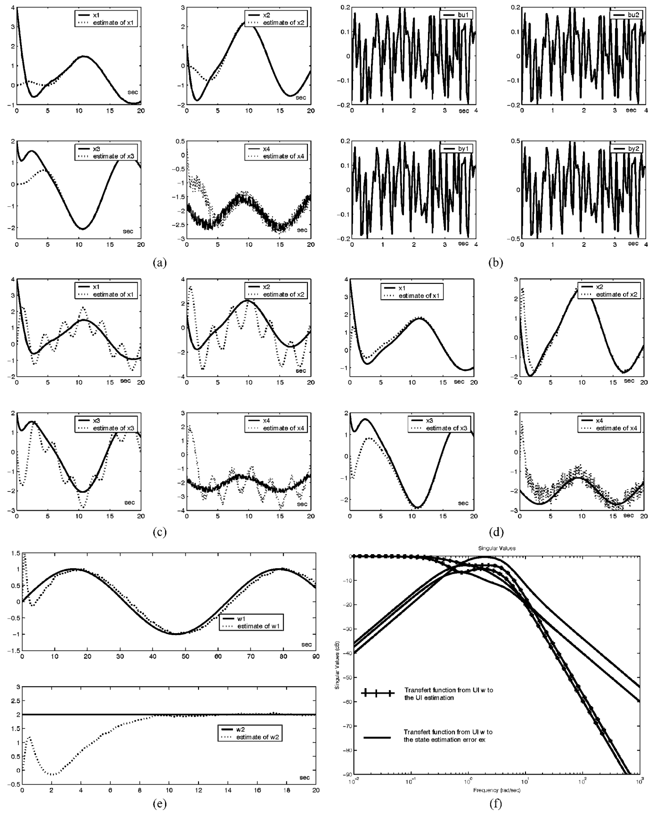

Fig. 1. State and UI estimation performance in presence of UI and actuator and sensor noises. (a) UIPO: state estimation where $\mathrm{wt} = \sin 2\mathrm{t}$ and $\mathrm{w2} = 2$. (b) Actuator and sensor noises. (c) PIO: state estimation where $\mathrm{w1} = \sin 2\mathrm{t}$ and $\mathrm{w2} = 25$.(d) PIO: state estimation where $\mathrm{w1} = \sin 0.2\mathrm{t}$ and $\mathrm{w2} = 2$. (e) PO: UI estimation where $\mathrm{w1} = \sin 0.1\mathrm{t}$ and $\mathrm{w2} = 2$. (f) PIO: transfer functions from w to $\hat{w}$ and w to e_x.

if there exists $P_1 > 0$ and U_1, such that

$$\begin{aligned} P_1\alpha_1 + \alpha_1^T P - U_1\beta_1 - \beta_1^T U_1^T + 2h_2 P_1 &< 0 \\ P_1\alpha_1 + \alpha_1^T P - U_1\beta_1 - \beta_1^T U_1^T + 2h_1 P_1 &> 0. \end{aligned} \tag{33}$$

Therefore, the convex optimization problem defined in Theorem 2 consists in finding P_1, U_1 and the maximal γ_1 subject to $P_1 > 0$, (33) and (12). After some iterations, we find $h_1 = 5.5$, $h_2 = 0.3$ and $\gamma_1 = 0.249$. Due to space limitation, the matrices U_1, P_1, Z_1, T, N, K_{p_1}, π and K_{p_2} are omitted. A satisfactory estimation is obtained for any UI and normally distributed random actuator and sensor noises. Fig. 1(a) shows that the state is well estimated.

B. PIO

Lemma 3 is satisfied for all p. In order to compare the estimation performances for both observers, the same LMI region D is defined (with $h_1 = 5.5$, $h_2 = 0.3$). There are several solutions Z_2 and we choose $Z_2 = [I_6\ 0]$ since it gives both maximal $rank T = n + m = 6$ and γ_1. The convex optimization problem defined in Theorem 4 consists in finding P_2, U_2 and the maximal γ_1 subject to $P_2 > 0$, (21) and

$$\begin{aligned} P_2 A_e + A_e^T P - U_2 C_e - C_e^T U_2^T + 2h_2 P_2 &< 0 \\ P_2 A_e + A_e^T P - U_2 C_e - C_e^T U_2^T + 2h_1 P_2 &> 0. \end{aligned}$$

After some iterations, we find $\gamma_1 = 0.2507$. Due to space limitation, the matrices $U_2, P_2, K_I, T, N, K_{p_1}, \pi$, and K_{p_2} are omitted. Contrary to the UIPO, the matrix Z_2 is not optimal. In fact, the designer must test different values for Z_2 until maximal rank T (i.e. $n + m = 6$) and γ_1 are obtained.

Satisfactory estimation is obtained for normally distributed random actuator and sensor noises. The observer gives a good UI estimation and Fig. 1(d) and (e) show that the state and UI are well estimated. More precisely, the UI attenuation properties can clearly be observed in the bode transfer function w to e_x given in Fig. 1(f) while the transfer w to $\hat{w}$ shows that the UI estimation error decreases at low frequencies. Fig. 1(c) shows a poor state estimation performance since the impact of the UI $w_1 = \sin 2t$ is not attenuated in this spectral domain (see Fig. 1(f)) whereas the UIPO presents a good estimation performance [see Fig. 1(a)]. Obviously, if we increase h_2 and h_1, we increase the bandwidth but we decrease the maximal Lipschitz constant γ_1. For example, with $h_1 = 20$, $h_2 = 1$ and $h_1 = 50$, and $h_2 = 2$, we find, respectively, $\gamma_1 = 0.172$ and $\gamma_1 = 0.049$. For $h_2 > 4$ the LMI constraints are infeasible.

Example 2: Consider the system described in [17] where $E = I_2$, $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, $F = G = 0$, $H = I_2$, $C = [0\ 1]$, $h_1 = 7$, and $h_2 = 6$. The convex optimization problem defined for the UIPO gives $\gamma_1 = 0.989$ although the Rajamani algorithm [17] gives only $\gamma_1 = 0.49$.

Example 3: For the system described in [24] where $H = I_4$, $F = G = 0$, $h_1 = 2.5$ and $h_2 = 0.95$, we obtain $\gamma_1 = 2.393$. Then, for $\gamma = 0.333$, our observer (3) is guaranteed to be exponentially stable since the Lipschitz constant is less than γ_1.

VI. Conclusion

We have presented a rigorous method for the design of observers for nonlinear descriptor systems in presence of UI and noise. Depending on the available knowledge on the dynamics of the UI, two cases were considered. First, without knowledge about the dynamics of the UI, an UIPO was proposed. Second, for UI with low frequencies, a PIO was proposed. Existence conditions of such observers have been given and proved with a strict LMI formulation.

Appendix

Proof of Lemma 1

Define the following nonsingular matrices V_1, V_2, and the full-column rank matrix V_3

$$V_1 = \begin{bmatrix} I_{n+m} & 0 \\ -\Theta_1^+\varphi_1 & I_{2(n+m+n_w)} \end{bmatrix},$$
$$V_3 = \begin{bmatrix} I_{n+m} & -\Psi_1\Theta_1^+ \\ 0 & I_{2n+5m} - \Theta_1\Theta_1^+ \\ 0 & \Theta_1\Theta_1^+ \end{bmatrix},$$
$$V_2 = \begin{bmatrix} -I_{n+m} & 0 & 0 & 0 & 0 \\ pI_{n+m} & I_{n+m} & 0 & 0 & 0 \\ 0 & 0 & I_{n+m} & 0 & 0 \\ 0 & 0 & 0 & -I_{n_w} & 0 \\ 0 & 0 & 0 & pI_{n_w} & I_{n_w} \end{bmatrix}.$$

Since

$$\begin{aligned} &\operatorname{rank}\begin{bmatrix} pI_{n+m} & \Psi_1 \\ \varphi_1 & \Theta_1 \end{bmatrix} - 2n - 3m - \operatorname{rank}G = n + n_w \\ &\Leftrightarrow \operatorname{rank}\begin{bmatrix} pI_{n+m} & \Psi_1 \\ \varphi_1 & \Theta_1 \end{bmatrix} V_2 - 2n - 3m - \operatorname{rank}G = n + n_w \\ &\Leftrightarrow \text{A4a} \end{aligned} \tag{34}$$

the problem of proving that A4a $\Leftrightarrow$ (20) is equivalent to prove that (34) $\Leftrightarrow$ (20).

Proof: (34) $\Leftrightarrow$ (20). From (17) $\Leftrightarrow$ A3a, we obtain

$$\begin{aligned}(34) \Leftrightarrow \text{rank} V_3 \begin{bmatrix} pI_{n+m} & \Psi_1 \\ \varphi_1 & \Theta_1 \end{bmatrix} V_1 - 2n - 3m - \text{rank} G \\ \Leftrightarrow n_w + \text{rank} \begin{bmatrix} pI_{n+m} - \Psi_1 \Theta_1^+ \varphi_1 \\ \left(I_{2n+5m} - \Theta_1 \Theta_1^+\right) \varphi_1 \end{bmatrix} - m \\ = n + n_w, \forall \mathbb{R}(p) \geq 0 \\ \Leftrightarrow (20)\end{aligned}$$

Proof of Lemma 2

Define the following full-rank matrix:

$$V_4 = \begin{bmatrix} T & 0 & N & 0 \\ 0 & I_{n_w} & 0 & 0 \\ 0 & 0 & 0 & I_m \\ \begin{bmatrix} 0 & 0 \\ 0 & pI_m \end{bmatrix} & 0 & \begin{bmatrix} I_m & 0 \\ -pI_m & I_m \end{bmatrix} & \begin{bmatrix} -pI_m \\ 0 \end{bmatrix} \end{bmatrix}$$

where $V_4 \in \mathbf{R}^{(n+m+n_w+3m)\times(n+m+n_w+3m)}$, $T \in \mathbf{R}^{(n+m)\times(n+m)}$, $\text{rank}T = n + m$ and $\text{rank}V_4 = n + m + n_w + 3m$ since $[T\ N]$ is of full-row rank, i.e., $n + m$. In addition, since

$$\text{rank} \begin{bmatrix} p\bar{E} - \bar{A} & -\bar{F} \\ 0 & pI_{n_w} \\ p\breve{C} & p\breve{G} \\ C_I & 0 \end{bmatrix} = n + m + \text{rank}\bar{F} \qquad \forall \mathbb{R}(p) \geq 0$$
$$\Leftrightarrow \text{A4b} \tag{35}$$

the problem of proving that A4b $\Leftrightarrow$ (29) is equivalent to prove that (35) $\Leftrightarrow$ (29). We obtain

$$\begin{aligned}(35) \Leftrightarrow rank V_4 \begin{bmatrix} p\bar{E} - \bar{A} & -\bar{F} \\ 0 & pI_{n_w} \\ p\breve{C} & p\breve{G} \\ C_I & 0 \end{bmatrix} \\ = n + m + n_w \qquad \forall \mathbb{R}(p) \geq 0 \\ \Leftrightarrow \text{rank} \begin{bmatrix} pI - T\bar{A} & -T\bar{F} + pN\breve{G} \\ 0 & pI_{n_w} \\ C_I & 0 \end{bmatrix} \\ = n + m + n_w \qquad \forall \mathbb{R}(p) \geq 0 \\ \Leftrightarrow (29).\end{aligned}$$

Note that all the previous equivalences hold using the Sylvester's inequality

$$\text{rank}\tilde{A} + \text{rank}\tilde{B} - m \leq \text{rank}\tilde{A}\tilde{B} \leq \min\{\text{rank}\tilde{A}, \text{rank}\tilde{B}\}$$

where $\tilde{A} \in \mathbb{R}^{n\times m}$, $\tilde{B} \in \mathbb{R}^{m\times p}$.

ACKNOWLEDGMENT

The author would like to thank the reviewers for their helpful comments.

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