in each step and X_{i+1} is set as $X_{i+1} = X_i + E_i$ until $\|\Omega(X_i)\|$ becomes less than a specified tolerance value. It may be noted that (27) is a Sylvester equation for E_i.

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Unknown Input Observers for Switched Nonlinear Discrete Time Descriptor Systems

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***Abstract*—In this paper, a linear matrix inequality technique for the state estimation of discrete-time, nonlinear switched descriptor systems is developed. The considered systems are composed of linear and nonlinear parts. An observer giving a perfect unknown input decoupled state estimation is proposed. Sufficient conditions of global convergence of observers are proposed. Numerical examples are given to illustrate this method.**

***Index Terms*—Hybrid systems, polyquadratic stability, switched descriptor systems, unknown input (UI) observers.**

I. INTRODUCTION

Switched control and/or observer systems have recently received much attention. Switched systems belong to a special class of hybrid systems. They are defined by a collection of dynamical (linear and/or nonlinear) subsystems together with a switching rule that specifies the switching between these subsystems. A survey on basic problems in switched system stability and design is available in [26] (see the references therein). Many such problems occur in practice: power converter systems where the switching signal is determined by pulse with pulsewidth modulation (PWM) and gain scheduling control systems are examples among many others. One can study the existence of a switching rule that ensures the stability of the switched system. One can assume that the switching sequence is not known *a priori*, and look for stability results under arbitrary switching sequences. On the one hand, most of the contributions in this field deal with stability analysis and control synthesis [7], [18]. On the other hand, unknown input observers (UIOs) have been widely studied for nonsingular systems [9], [29], singular systems [6], [10], [16], nonlinear descriptor systems [17], and recently, for switched nonsingular systems [20]. Nevertheless, there is no result extending the method mentioned in [20] to the general representation of switched nonlinear descriptor systems, although many practical systems can be described by them [2], and their fault diagnosis may be based on UIO design [21].

As mentioned in [32], there are generally two broad approaches for a nonlinear observer design. In the first approach, the objective is to find a coordinate transformation so that the state-estimation error dynamics are linear in the new coordinates, and then, linear techniques can be performed [13], [14], [30]. Necessary and sufficient conditions have been established [19], [30] for the existence of such a coordinate transformation. The second approach does not need the transformation, and the observer design is directly based on the original sys-

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tem. Because of the complexity of nonlinear systems, a lot of directly designing methods have been developed. For instance, Praly *et al.* [15], [22], [28] contributed some results on an observer design using high-gain techniques. Besançon and Hammouri [3] and Dawson [12] studied the observer design from the solution of the Riccati equation for Lipschitz nonlinear systems. Adaptive observers have been proposed for special classes of nonlinear systems [5], [23]. For the class of global Lipschitz nonlinear systems, the existence condition has been established for a full-order observer and also for reduced-order observers, respectively, in [24] and [32]. The design method is based on the solution of a Riccati equation. More recently, based on the linear matrix inequality (LMI) approach both the proportional and proportional integral observer for the nonlinear descriptor system have been proposed in [17]. According to [17, Remark 1], the nonlinear systems considered in this paper are more general than that in [5], [24], [32]. Moreover, we proposed to extend the design of a proportional observer for an unsquare (rectangular) switched descriptor system that includes both UI and Lipschitz nonlinearities. The systems considered are also in a general form and seem to be the first using convex optimization. Briefly, an extension of the UIO design for a linear system to a nonlinear system is proposed.

This paper is organized as follows. Section II presents the problem statement. A design method of the proportional observer and the main results of this note are given in Section III. In Section IV, the performance of the proportional switched observer is evaluated through two numerical examples. The proof of the detectability condition is provided in the Appendix. Finally, Section V concludes the paper.

II. Problem Formulation

Consider the switched nonlinear descriptor systems

$$\begin{aligned} E_{\alpha(k+1)}x_{k+1} &= A_{\alpha(k)}x_k + F_{\alpha(k)}d_k + H_{\alpha(k)}\phi_k \\ y_k &= C_{\alpha(k)}x_k + G_{\alpha(k)}d_k \end{aligned} \tag{1}$$

where $E_{\alpha(k+1)}$ and $A_{\alpha(k)} \in \mathbb{R}^{p\times n}$ are in the general form and may be rectangular, $F_{\alpha(k)} \in \mathbb{R}^{p\times q}$, $H_{\alpha(k)} \in \mathbb{R}^{p\times n_\phi}$, $C_{\alpha(k)} \in \mathbb{R}^{m\times n}$, $G_{\alpha(k)} \in \mathbb{R}^{m\times q}$, $p \le n$, $x \in \mathbb{R}^n$, $d \in \mathbb{R}^q$, $\phi_k = \phi(x_k, u_k, k)$: $\mathbb{R}^n \times \mathbb{R}^{n_u} \times \mathbb{N} \to R^{n_\phi}$, and $y \in \mathbb{R}^m$ denote, respectively, the descriptor vector, the unknown input vector, the nonlinearity vector and the output vector. In the sequel, disturbances or partial inputs that are inaccessible are called unknown inputs. The signal $u \in \mathbb{R}^{n_u}$ is the control input vector. The variable $\alpha(k)$ is a piecewise constant switching signal taking value from the finite index set $\varepsilon = \{1, 2, \dots, h\}$. At a switching time k, we have $\alpha(k-1) \neq \alpha(k)$. The ordered sequence of the switching times is said to be the switching time sequence of the switching signal. It is assumed that the switching time sequence is real-time accessible, depending on the control input or on the measured output, or using a finite automation or any strategy. The family of matrices $\{(E_i, A_i, F_i, H_i, C_i, G_i) : i \in \varepsilon\}$ are parameterized by an index set $\varepsilon = \{1, 2, \dots, h\}$ and $i = \alpha(k)$. Moreover, $\alpha(k) = i$ and $\alpha(k+1) = j$ means that the matrices $(E_j, A_i, F_i, H_i, C_i, G_i, D_i)$ are activated.

Notation 1: $(\cdot)^T$ stands for the transpose matrix, $(*)$ is used for the blocks induced by symmetry, $(\cdot) > 0$ denotes a symmetric positive definite matrix, $(\cdot)^+$ is the pseudoinverse matrix, $(\cdot)^\perp$ is the orthogonal complement, $||\cdot||$ stands for the Euclidean norm, and $(\cdot)_{k_+}$ stands for $(.)_{\alpha(k),\alpha(k+1)}$, for instance, $T_{k_+} = T_{\alpha(k),\alpha(k+1)}$.

Remark 1: The orthogonal complement $A^\perp$ for a real $n \times p$ matrix A with rank q is defined as an $(n-q)\times n$ matrix such that $AA^\perp = 0$ and $A^\perp A^{\perp T} > 0$.

Assumptions: In the sequel, the following are assumed.

The nonlinearity $\phi(x_k, u_k, k)$ is globally Lipschitz in x with Lipschitz constant γ, i.e.

$$\text{A1} \quad \|\phi(x_k, u_k, k) - \phi(\hat{x}_k, u_k, k)\| \le \gamma \|x_k - \hat{x}_k\| \quad \forall u \in \mathbb{R}^{n_u}, \quad k \in \mathbb{N}.$$

For instance, the sinusoidal terms usually encountered in many problems of robotics are all global Lipschitz. Moreover, most nonlinearities are local Lipschitz if they are considered in a given neighborhood (see [23, def.])

$$\text{A2} \quad \begin{cases} \text{rank}\begin{bmatrix} E_{\alpha(k+1)} & F_{\alpha(k)} & 0 \\ 0 & G_{\alpha(k)} & 0 \\ C_{\alpha(k+1)} & 0 & G_{\alpha(k+1)} \end{bmatrix} \\ = n + \text{rank}\, G_{\alpha(k+1)} + \text{rank}\begin{bmatrix} F_{\alpha(k)} \\ G_{\alpha(k)} \end{bmatrix} \end{cases}$$

$$\text{A3} \quad \begin{cases} \text{rank}\begin{bmatrix} zE_i - A_i & -F_i \\ C_i & G_i \end{bmatrix} = n + \text{rank}\begin{bmatrix} F_i \\ G_i \end{bmatrix} \\ = \forall |z| \ge 1, \quad i \in \varepsilon \end{cases}$$

$$\text{A4} \quad \begin{cases} p + 2m > n + q + \text{rank}\, G_{\alpha(k)} \\ \text{rank}\begin{bmatrix} F_{\alpha(k)} \\ G_{\alpha(k)} \end{bmatrix} = q \\ \text{rank}\begin{bmatrix} C_{\alpha(k)} & G_{\alpha(k)} = m \end{bmatrix}. \end{cases}$$

Remark 2: Define

$$V_1 = \begin{bmatrix} I_n & 0 & 0 \\ C_{\alpha(k+1)} & 0 & -I_m \\ 0 & I_m & 0 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & -I_q \end{bmatrix}$$

$$\Gamma = \begin{bmatrix} I_n & F_{\alpha(k)} & 0 \\ 0 & G_{\alpha(k)} & 0 \\ C_{\alpha(k+1)} & 0 & G_{\alpha(k+1)} \end{bmatrix}.$$

For $E_{\alpha(k+1)} = I_n$, the assumption A2 becomes equivalent to [11, Assumption (12)], since

$$\text{rank}\,\Gamma = n + \text{rank}\, G_{\alpha(k+1)} + \text{rank}\begin{bmatrix} F_{\alpha(k)} \\ G_{\alpha(k)} \end{bmatrix}$$

is equivalent to

$$\begin{aligned} \text{rank}\, V_1\Gamma V_2 &= \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & -I_q \end{bmatrix} \\ &= n + \text{rank}\, G_{\alpha(k+1)} + \text{rank}\begin{bmatrix} F_{\alpha(k)} \\ G_{\alpha(k)} \end{bmatrix} \end{aligned}$$

which is equivalent to

$$\begin{aligned} \text{rank}&\begin{bmatrix} I_n & F_{\alpha(k)} & 0 \\ 0 & C_{\alpha(k+1)}F_{\alpha(k)} & G_{\alpha(k+1)} \\ 0 & G_{\alpha(k)} & 0 \end{bmatrix} \\ &= n + \text{rank}\, G_{\alpha(k+1)} + \text{rank}\begin{bmatrix} F_{\alpha(k)} \\ G_{\alpha(k)} \end{bmatrix} \end{aligned}$$

which is equivalent to [11, eq. (12)]. In addition, for $\alpha(k+1)=\alpha(k)$, the assumption A2 becomes equivalent to [29, condition (1–1)].

Remark 3: The assumptions

$$\text{rank}\begin{bmatrix} F_{\alpha(k)} \\ G_{\alpha(k)} \end{bmatrix} = q$$
$$\text{rank}[C_{\alpha(k)} \quad G_{\alpha(k)}] = m$$

ensure, respectively, that the UIs and measurements are linearly independent. This can always be satisfied by redefining the UI and the measurement vector [10]. While, according to Remark 1, $p+2m > n+q+\text{rank}\, G_{\alpha(k)}$ is necessary in order to ensure that $\Theta_{k_+}^{\perp}$ is well defined.

Our aim is to design an observer in the form

$$\begin{aligned} z_{k+1} &= \Pi_{k_+} z_k + K_{k_+} y_k + T_{k_+} H_{\alpha(k)} \phi(\hat{x}_k, u_k, k) \\ \hat{x}_k &= z_k + N_{\alpha(k-1),\alpha(k)} y_k \end{aligned} \tag{2}$$

where $z_k \in \mathbb{R}^n$ and

$$[T_{k_+} \quad N_{k_+} \quad K_{1k_+} \quad \Pi_{k_+}] = \Psi\Theta_{k_+}^{+} - Z_{\alpha(k)}\Theta_{k_+}^{\perp}$$

with

$$\Theta_{k_+} = \begin{bmatrix} E_{\alpha(k+1)} & A_{\alpha(k)} & F_{\alpha(k)} & 0 \\ C_{\alpha(k+1)} & 0 & 0 & G_{\alpha(k+1)} \\ 0 & -C_{\alpha(k)} & -G_{\alpha(k)} & 0 \\ 0 & -I_n & 0 & 0 \end{bmatrix}$$

$$\Psi = [I_n \quad 0_{n\times(n+2q)}],\ \Theta_{k_+}^{\perp} = (I_{n+p+2m} - \Theta_{k_+}\Theta_{k_+}^{+})$$
$$K_{k_+} = K_{1k_+} + \Pi_{k_+} N_{\alpha(k-1),\alpha(k)}.$$

The problem of the observer design is also reduced to finding matrices $Z_{\alpha(k)}$ such that the estimate $\hat{x}_k$ converges asymptotically to the state x_k.

III. Observer Design

In this section, a new method is presented to design the observer (2) for a switched nonlinear system (1). The following theorem will give the structure of the observer.

Theorem 1: Under A2, there exist matrices T_{k_+}, N_{k_+}, K_{1k_+}, and Π_{k_+} such that

$$T_{k_+} E_{\alpha(k+1)} + N_{k_+} C_{\alpha(k+1)} = I_n \tag{3}$$
$$\Pi_{k_+} = T_{k_+} A_{\alpha(k)} - K_{1k_+} C_{\alpha(k)} \tag{4}$$
$$T_{k_+} F_{\alpha(k)} - K_{1k_+} G_{\alpha(k)} = 0 \tag{5}$$
$$N_{k_+} G_{\alpha(k+1)} = 0 \tag{6}$$

and the difference of the state-estimation error $e_k = x_k - \hat{x}_k$ becomes

$$e_{k+1} = \Pi_{k_+} e_k + T_{k_+} H_{\alpha(k)} \tilde{\phi}_k \tag{7}$$

where

$$\tilde{\phi}_k = \phi(x_k, u_k, k) - \phi(\hat{x}_k, u_k, k) \tag{8}$$
$$K_{k_+} = K_{1k_+} + \Pi_{k_+} N_{\alpha(k-1),\alpha(k)} \tag{9}$$

Remark 4: Consider the single system (1) where $\varepsilon = \{1\}$, $\alpha(k+1) = \alpha(k+1) = 1$, $E_{\alpha(k+1)} = E_1$, $A_{\alpha(k)} = A_1$, $F_{\alpha(k)} = F_1$, $H_{\alpha(k)} = 0$, $C_{\alpha(k)} = C_1$ and $G_{\alpha(k)} = G_1$. When G_1 has a full row rank, the matrix C_{12} defined in [10] is necessarily equal to zero. Consequently, the matrices N and M defined by [10, eqs. (24) and (25)] cannot be computed, and the observer is unfeasible. Furthermore, in our approach, when G_1 has a full row rank, it follows that the only $N_{1,1}$ that fulfills $N_{1,1}G_1 = 0$ is the zero matrix. So, the observer (2) is solvable, provided the matrix $\Pi_{1,1} = T_{1,1}A_1 - K_{1,1}C_1$ is stable, $T_{1,1}E_1 = I_n$, and $K_{1_{1,1}}G_1 = T_{1,1}F_1$. In other words, E_1 must be nonsingular ($T_{1,1} = E_1^{-1}$) and the row image of $E_1^{-1}F_1$ has to be included in the row image of G_1, while the solution $K_{1,1}$ of $K_{1_{1,1}}G_1 = E_1^{-1}F_1$ must ensure the stability of $\Pi_{1,1} = E_1^{-1}A_1 - K_{1,1}C_1$. This is very restrictive, but a solution may exist. So, our observer may exist even if the number of UI in the measurement equation is equal to the number of the measurement. In addition, the detectability condition A3 is the usual condition defined in UIO theory; see, for instance, in [10, eq. (23)]. So, the methodologies proposed are no less restrictive than those reported in the literature [6], [8], [10], [11], [17], [29].

Proof: Suppose that (3) holds, then $e_{k+1} = x_{k+1} - \hat{x}_{k+1}$ becomes

$$e_{k+1} = T_{k_+} E_{\alpha(k+1)} x_{k+1} - z_{k+1} - N_{k_+} G_{\alpha(k+1)} d_{k+1}$$

and from (1), (2), and (8), e_{k+1} becomes

$$\begin{aligned} e_{k+1} &= (T_{k_+} A_{\alpha(k)} - \Pi_{k_+} T_{k_+} E_{\alpha(k)} - K_{k_+} C_{\alpha(k)}) x_k \\ &\quad + \Pi_{k_+} e_k + T_{k_+} H_{\alpha(k)} \tilde{\phi}_k - N_{k_+} G_{\alpha(k+1)} d_{k+1} \\ &\quad + (T_{k_+} F_{\alpha(k)} - (K_{k_+} - \Pi_{k_+} N_{\alpha(k-1),\alpha(k)}) G_{\alpha(k)}) d_k. \end{aligned} \tag{10}$$

Substituting (9) into (10) and using the constraints (4)–(6), $T_{\alpha(k-1),\alpha(k)} E_{\alpha(k)} + N_{\alpha(k-1),\alpha(k)} C_{\alpha(k)} = I_n$, (7) is obtained. Rewriting (7) and (3)–(6), respectively, leads to

$$\begin{aligned} e_{k+1} &= [T_{k_+} \quad N_{k_+} \quad K_{1k_+} \quad \Pi_{k_+}] \varphi_{1\alpha(k)} e_k \\ &\quad + [T_{k_+} \quad N_{k_+} \quad K_{1k_+} \quad \Pi_{k_+}] \varphi_{2\alpha(k)} \tilde{\phi}_k \end{aligned} \tag{11}$$
$$\Psi = [T_{k_+} \quad N_{k_+} \quad K_{1k_+} \quad \Pi_{k_+}] \Theta_{k_+} \tag{12}$$

where

$$\varphi_{1\alpha(k)} = \begin{bmatrix} A_{\alpha(k)} \\ 0_{m\times n} \\ -C_{\alpha(k)} \\ 0_{n\times n} \end{bmatrix}, \quad \varphi_{2\alpha(k)} = \begin{bmatrix} H_{\alpha(k)} \\ 0_{m\times n} \\ 0_{m\times n} \\ 0_{n\times n} \end{bmatrix}$$

The solution $[T_{k_+} \quad N_{k_+} \quad K_{1k_+} \quad \Pi_{k_+}]$ of (12) depends on the rank of matrix Θ_{k_+}. A solution exists if and only if [25]

$$\text{rank}\begin{bmatrix} \Theta_{k_+} \\ \Psi \end{bmatrix} = \text{rank}\, \Theta_{k_+} \tag{13}$$

which is equivalent to A2. Therefore, under A2, the general solution of (12) is

$$[T_{k_+} \quad N_{k_+} \quad K_{1k_+} \quad \Pi_{k_+}] = \Psi\Theta_{k_+}^{+} - Z_{\alpha(k)}\Theta_{k_+}^{\perp} \tag{14}$$

where $\Theta_{k_+}^{\perp} = (I_{n+p+2m} - \Theta_{k_+}\Theta_{k_+}^{+})$ and $Z_{\alpha(k)}$ is an arbitrary matrix of appropriate dimension.

Substituting (14) into (11) gives (7), where Π_{k_+} and T_{k_+} are determined by known matrices and by the arbitrary matrix $Z_{\alpha(k)}$ as follows:

$$\Pi_{k_+} = \Psi\Theta_{k_+}^{+}\varphi_{1\alpha(k)} - Z_{\alpha(k)}\Theta_{k_+}^{\perp}\varphi_{1\alpha(k)} \tag{15}$$
$$T_{k_+} H_{\alpha(k)} = \Psi\Theta_{k_+}^{+}\varphi_{2\alpha(k)} - Z_{\alpha(k)}\Theta_{k_+}^{\perp}\varphi_{2\alpha(k)}. \tag{16}$$

■

Now, the condition of global stability of (7) is stated in the following theorem.

Theorem 2: If there exist symmetric positive definite matrices $P_1, P_2, \ldots, P_h$ and matrices $U_1, U_2, \ldots, U_h$ satisfying

$$\begin{bmatrix} P_i + P_i^T - P_j & X_1 & X_2 & 0 \\ * & P_i & 0 & \gamma I_n \\ * & * & I_n & 0 \\ * & * & * & I_n \end{bmatrix} > 0 \quad \forall i, j \in \varepsilon, \tag{17}$$

then the state-estimation error e_k converges globally toward the origins $X_{1,2} = P_i \Psi \Theta_{i,j}^+ \varphi_{1i} - U_i \Theta_{i,j}^\perp \varphi_{1i}$ and $X_2 = P_i \Psi \Theta_{i,j}^+ \varphi_{2i} - U_i \Theta_{i,j}^\perp \varphi_{2i}$. Moreover, the resulting observer gains are given by (9) and (14), where the matrices Z_i's are given by $Z_i = P_i^{-1} U_i$.

Proof: Consider the switched Lyapunov function $V(e_k, k) = e_k^T P_{\alpha(k)} e_k$ where $P_{\alpha(k)} > 0$ is a positive definite matrix. If such a Lyapunov function exists, and its difference $\Delta V = V(e_{k+1}, k+1) - V(e_k, k)$ is negative definite along system trajectories of (7), then the origin of the system (7) is globally asymptotically stable. By computing the difference ΔV along the solution of (7), ΔV is given by

$$\begin{aligned} \Delta V &= e_{k+1}^T P_{\alpha(k+1)} e_{k+1} - e_k^T P_{\alpha(k)} e_k \\ &= e_k^T \Pi_{k_+}^T P_{\alpha(k+1)} \Pi_{k_+} e_k - e_k^T P_{\alpha(k)} e_k + 2 e_k^T \Pi_{k_+}^T P_{\alpha(k+1)} T_{k_+} \\ &\quad \times H_{\alpha(k)} \tilde{\phi}_k + \tilde{\phi}_k^T H_{\alpha(k)}^T T_{k_+}^T P_{\alpha(k+1)} T_{k_+} H_{\alpha(k)} \tilde{\phi}_k \\ &\leq e_k^T \Pi_{k_+}^T P_{\alpha(k+1)} \Pi_{k_+} e_k + 2 e_k^T \Pi_{k_+}^T P_{\alpha(k+1)} T_{k_+} H_{\alpha(k)} \tilde{\phi}_k \\ &\quad + \tilde{\phi}_k^T H_{\alpha(k)}^T T_{k_+}^T P_{\alpha(k+1)} T_{k_+} H_{\alpha(k)} \tilde{\phi}(k) \\ &\quad - e_k^T P_{\alpha(k)} e_k - \tilde{\phi}_k^T \tilde{\phi}_k + \gamma^2 e_k^T e_k \end{aligned}$$

since, from A1 and (8), we have $-\tilde{\phi}_k^T \tilde{\phi}_k + \gamma^2 e_k^T e_k \geq 0$.

Now, ΔV can be written as

$$\Delta V(e_k, k) \leq e_{a_k}^T \begin{bmatrix} \Gamma_{k_+} & \Pi_{k_+}^T P_{\alpha(k+1)} T_{k_+} H_{\alpha(k)} \\ * & H_{\alpha(k)}^T T_{k_+}^T P_{\alpha(k+1)} T_{k_+} H_{\alpha(k)} - I_{n_\phi} \end{bmatrix} e_{a_k}$$

where $\Gamma_{k_+} = \Pi_{k_+}^T P_{\alpha(k+1)} \Pi_{k_+} - P_{\alpha(k)} + \gamma^2 I_n$ and $e_{a_k}^T = [e_k^T \quad \tilde{\phi}_k^T]$. The difference $\Delta V(e_k, k)$ is negative definite for any $[e_k^T \quad \tilde{\phi}_k^T] \neq 0$ if

$$\begin{bmatrix} \Gamma_{k_+} & \Pi_{k_+}^T P_{\alpha(k+1)} T_{k_+} H_{\alpha(k)} \\ * & H_{\alpha(k)}^T T_{k_+}^T P_{\alpha(k+1)} T_{k_+} H_{\alpha(k)} - I_{n_\phi} \end{bmatrix} < 0. \tag{18}$$

As this inequality has to be satisfied under the arbitrary switching law, it follows that it should hold for special configuration $\alpha(k+1) = j$ and $\alpha(k) = i$. Define $X_3 = P_i - \Pi_{i,j}^T P_j \Pi_{i,j} - \gamma^2 I_n$, and then (18) becomes

$$\begin{bmatrix} X_3 & -\Pi_{i,j}^T P_j T_{i,j} H_i \\ * & -H_i^T T_{i,j}^T P_j T_{i,j} H_i + I_{n_\phi} \end{bmatrix} > 0 \quad \forall i, j \in \varepsilon \tag{19}$$

which is equivalent, by Schur complement, to

$$\left[\begin{array}{c|cc} P_j & P_j \Pi_{i,j} & P_j T_{i,j} H_i \\ \hline * & P_i - \gamma^2 I_n & 0 \\ * & * & I_{n_\phi} \end{array}\right] > 0 \quad \forall i, j \in \varepsilon$$

which is equivalent, by Schur complement, to

$$\left[\begin{array}{ccc|c} P_j & P_j \Pi_{i,j} & P_j T_{i,j} H_i & 0 \\ * & P_i & 0 & \gamma I_n \\ * & * & I_{n_\phi} & 0 \\ \hline * & * & * & I_n \end{array}\right] > 0 \quad \forall i, j \in \varepsilon$$

which is equivalent by [31, Lemma 1] to

$$\left[\begin{array}{c|ccc} P_i + P_i^T - P_j & P_i \Pi_{i,j} & P_i T_{i,j} H_i & 0 \\ \hline * & P_i & 0 & \gamma I_n \\ * & * & I_{n_\phi} & 0 \\ * & * & * & I_n \end{array}\right] > 0 \quad \forall i, j \in \varepsilon \tag{20}$$

where the matrices S, M, Q, and G in [31] are directly identified by $S = P_j$, $M^T = [\Pi_{i,j} \quad T_{i,j} H_i \quad 0]$, $G = P_i$, and

$$Q = \begin{bmatrix} P_i & 0 & \gamma I_n \\ * & I_{n_\phi} & 0 \\ * & * & I_n \end{bmatrix}.$$

Substituting (16) and $U_i = P_i Z_i$ into (20) with $\alpha(k) = i$ and $\alpha(k+1) = j$, (17) is obtained. ■

Remark 5: The feasibility of (17), or equivalently, of (19), implies that the pairs $(\Psi \Theta_{i,i}^+ \varphi_{1i},\ \Theta_{i,i}^\perp \varphi_{1i})$ are detectable. Indeed, according to Theorem 2, satisfying (17) is equivalent to guarantee the stability of the state-estimation error (7), whatever the switching rule may be. This includes the case where the switching rule leads to a linear behavior, i.e., $\alpha(k+1) = \alpha(k) = i$ and $\Pi_{i,i}$ has to be Hurwitz. In other words, for $\alpha(k+1) = \alpha(k) = i$, the existence of a solution $P_i > 0$, U_i of the LMI (17) needs that the matrix $\Pi_{i,i} = \Psi \Theta_{i,i}^+ \varphi_{1i} - Z_i \Theta_{i,i}^\perp \varphi_{1i}$ is Hurwitz (in the meaning of Lyapunov stability), since the element $(1, 1)$ of (19) implies $-P_i + \Pi_{i,i}^T P_i \Pi_{i,i} < -\gamma^2 I_{n+m} < 0$.

Remark 6: Of course, the switched detectability of the system (7) is not ensured by the assumption that, for each subsystem $i \in \varepsilon$, the pair $(\Psi \Theta_{i,i}^+ \varphi_{1i},\ \Theta_{i,i}^\perp \varphi_{1i})$ is detectable (see the example in [4, Sec. 7.2]), but the detectability of each pair $(\Psi \Theta_{i,i}^+ \varphi_{1i},\ \Theta_{i,i}^\perp \varphi_{1i})$ is a necessary condition to solve (17). Moreover, an arbitrary choice of $T_{i,i}$ can involve a loss of detectability of the pair $(\Psi \Theta_{i,i}^+ \varphi_{1i},\ \Theta_{i,i}^\perp \varphi_{1i})$ (see [9]). To overcome the problem of an arbitrary choice of $T_{i,i}$, the computation of a suitable $T_{i,i}$ is included in the design procedure. That is why (7) is rewritten as (11), where $[T_{k_+} \quad N_{k_+} \quad K_{1k_+} \quad \Pi_{k_+}]$ is given by (14). Thus, the matrix Z_i involved in $T_{i,i}$ (14) plays the role of a parametrization. The switched observer design is finally reduced to the computation of the gain matrices Z_i, $i \in \varepsilon$, ensuring the asymptotic stability of system (7) under arbitrary switching signal.

Now, the following results can be established.

Lemma 1: There exist matrices Z_i such that the matrices $\Pi_{i,i} = \Psi \Theta_{i,i}^+ \varphi_{1i} - Z_i \Theta_{i,i}^\perp \varphi_{1i}$ are Hurwitz if and only if the pair $(\Psi \Theta_{i,i}^+ \varphi_{1i},\ \Theta_{i,i}^\perp \varphi_{1i})$ is detectable, which is equivalent to (21), which is equivalent to A3.

$$\operatorname{rank} \begin{bmatrix} z I_n - \Psi \Theta_{i,i}^+ \varphi_{1i} \\ \Theta_{i,i}^\perp \varphi_{1i} \end{bmatrix} = n \quad \forall\ |z| \geq 1. \tag{21}$$

Proof: See the Appendix. ■

IV. Examples

In this section, the results are illustrated with two simulations. In the first one, the studied system is nonsingular, and it is derived from the continuous system of [24], while the second simulation concerns a switched systems subject to UI, nonlinearities, and algebraic constraints.

Example 1: From [24], the observer (2) for system (1a,1b) is guaranteed to be stable for all nonlinearities with Lipschitz constant of magnitude less than 0.49. Using the aforementioned LMI formulation, it is proposed to find the largest Lipschitz constant γ such that the observer (2) exists for system (1). The system (1a,1b) considered in [24] is first approximated by the Euler approximation, where, for a good approximation, the sample time is fixed to $T_e = 0.01\ s$. Let us consider the discrete-time model (1), where $\varepsilon = \{1\}$, $\alpha(k) = 1\ \forall k$, $E_1 = I$, $A_1 = I_2 + T_e\bar{A}$, $F_1 = 0$, $G_1 = 0$, $H_1 = T_e I_2$, $C_1 = [0 \quad 1]$, and $\bar{A} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right]$.

It is assumed that the nonlinearity $\phi(x_k, u_k, k)$ is globally Lipschitz in x_k with Lipschitz constant γ, and A2 holds since $E = I_2$ and A3 holds for all z.

For a known Lipschitz constant γ, Theorem 2 gives the gain observer Z_1 such that observer (2) for system (1) exists. Theorem 2 can be reformulated as the following convex optimization problem

$$\max_{P_1, U_1} \quad \gamma \text{ subject to (17) and } P_1 = P_1^T > 0 \tag{22}$$

where $i = j = 1$

$$\Theta_{k_+} = \begin{bmatrix} I_2 & A_1 \\ C_1 & 0 \\ 0 & -C_1 \\ 0 & -I_2 \end{bmatrix}$$

and $\Psi = [\, I_n \quad 0_{n\times n} \,]$. Applying the convex optimization problem defined by (22), the following results are obtained: $\gamma = 0.9950$

$$T_{1,1} = \begin{bmatrix} 1 & -9.99 \\ 0 & 0.0141 \end{bmatrix}$$

$$N_{1,1} = \begin{bmatrix} 9.99 \\ 0.9859 \end{bmatrix}$$

$$K_{1,1} = \begin{bmatrix} -0.8881 \\ 0.0152 \end{bmatrix}$$

$$\Pi_{1,1} = \begin{bmatrix} 0.9001 & 0 \\ 0.0001 & 0.0047 \end{bmatrix}.$$

It can be noted that the maximal constant of Lipschitz obtained by the present approach is larger than the Lipschitz constant given by [24]. If $C_1 = [1 \quad 0]$, the maximal constant of Lipschitz is $\gamma = 1.4142$. The following example shows that a switched observer may exist for a more general class.

Example 2: Consider the switched nonlinear descriptor systems (1), where

$$E_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad F_i = \begin{bmatrix} f_{11_i} & 0 \\ 0 & f_{22_i} \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A_i = \begin{bmatrix} -1 & a_{12_i} & 0 & a_{14_i} \\ -1 & 0 & 0 & 1 \\ 0 & -1 & a_{33_i} & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad x_k = \begin{bmatrix} x_{1k} \\ x_{2k} \\ x_{3k} \\ x_{4k} \end{bmatrix}$$

$$H_i = \begin{bmatrix} 1 \\ 0 \\ h_{31_i} \\ 0 \end{bmatrix}, \quad \phi_k = \gamma \sin x_{1k}, \quad C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & c_{23_i} & 1 \\ 0 & 0 & 0 & c_{34_i} \end{bmatrix},$$

$$G_i = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad d_k = \begin{bmatrix} d_{1k} \\ d_{2k} \end{bmatrix}, \quad \varepsilon = \{1, 2\}, \gamma = 0.5,$$

$$T_e = 0.01 \text{ s}, d_{1k} = \sin 4kT_e, d_{2k} = \sin 0.1kT_e$$

$$a_{12_1} = 0.4, a_{12_2} = 0.6, a_{33_1} = -0.4, a_{33_2} = -0.6$$

$$a_{14_1} = 0.2, a_{14_2} = 0, c_{23_1} = 1, c_{23_2} = 0, c_{34_1} = 1, c_{34_2} = 0$$

$$h_{31_1} = 1, h_{31_2} = 0, f_{11_1} = 0, f_{11_2} = 1, f_{22_1} = 1, f_{22_2} = 0$$

and where the switching time sequence is given by Table I.

TABLE I
SWITCHING SEQUENCE

k	0	...	49	50	51	52	...	250	251	252	253	...
$\alpha(k-1)$	2	...	2	2	2	1	...	1	1	1	2	...
$\alpha(k)$	2	...	2	2	1	1	...	1	1	2	2	...
$\alpha(k+1)$	2	...	2	1	1	1	...	1	2	2	2	...

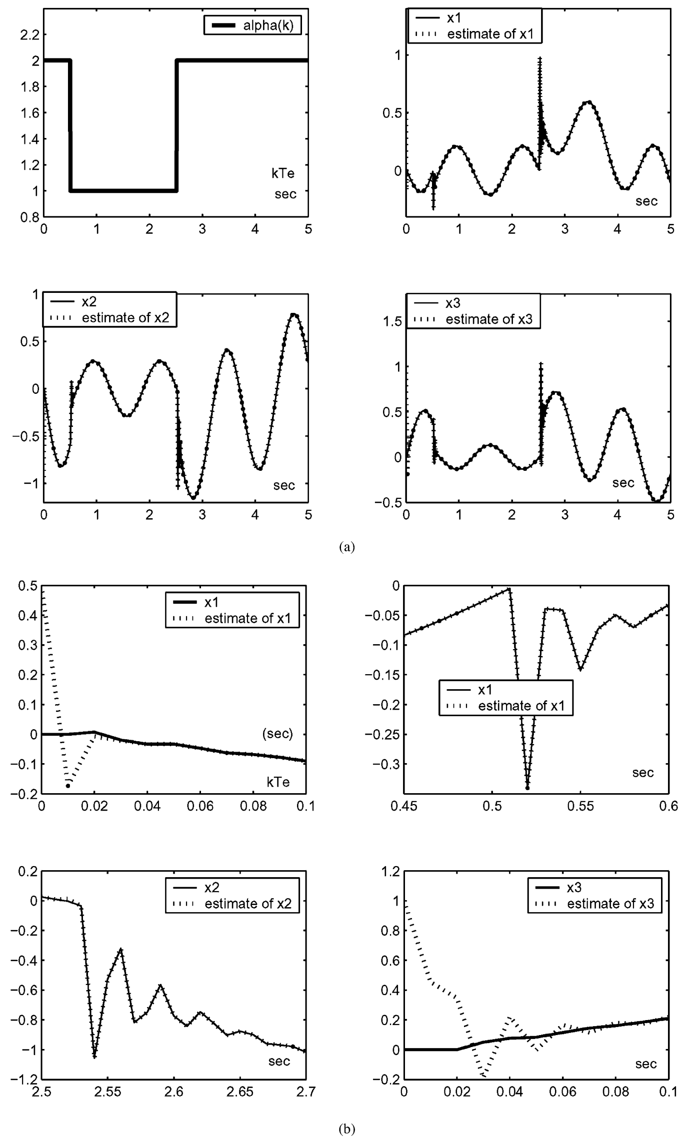

Fig. 1. Switching time sequence and state estimation performance. (a) Switching time sequence and state estimation. (b) Zoom of the state estimation.

Remark 7: If the switching time sequence is unknown *a priori*, a switching rule can be defined, for instance, see [20, example 1].

Algorithm

1) The assumption A1 holds for $\gamma = 0.5$. Assumption A2 holds for all couples $\{(2,2);\ (2,1);\ (1,1);\ (2,2)\}$, for instance, $\alpha(k+1) = 2$, $\alpha(k) = 1$, and the equality

$$\begin{bmatrix} E_2 & F_1 & 0 \\ 0 & G_1 & 0 \\ C_2 & 0 & G_2 \end{bmatrix} = n + \text{rank } G_2 + \text{rank} \begin{bmatrix} F_1 \\ G_1 \end{bmatrix}$$

is satisfied. The assumption A3 holds for all $|z| \geq 1$ and for all $i \in \varepsilon = \{1, 2\}$.

2) From Table I, $\varphi_{1_1}, \varphi_{2_1}, \varphi_{1_2}, \varphi_{2_2}, \Theta_{2,2}, \Theta_{2,1}, \Theta_{1,1}, \Theta_{1,2}$, and Ψ are computed. Since assumptions A1–A4 hold, one can solve the convex optimization problem defined in Theorem 2. More precisely, finding P_1, P_2, U_1, U_2 subject to $P_1 = P_1^T > 0, P_2 = P_2^T > 0$, (17) with $i,j = 2,2$, (17) with $i,j = 2,1$, (17) with $i,j = 1,1$, and (17) with $i,j = 1,2$. After 26 iterations, the gains Z_1 and Z_2 are obtained. From (14), we deduce

$$[T_{2,2} \quad N_{2,2} \quad K_{1_{2,2}} \quad \Pi_{2,2}] = \Psi\Theta_{2,2}^{+} - Z_2\Theta_{2,2}^{\perp}$$
$$[T_{2,1} \quad N_{2,1} \quad K_{1_{2,1}} \quad \Pi_{2,1}] = \Psi\Theta_{2,1}^{+} - Z_2\Theta_{2,1}^{\perp}$$
$$[T_{1,1} \quad N_{1,1} \quad K_{1_{1,1}} \quad \Pi_{1,1}] = \Psi\Theta_{1,1}^{+} - Z_1\Theta_{1,1}^{\perp}$$
$$[T_{1,2} \quad N_{1,2} \quad K_{1_{1,2}} \quad \Pi_{1,2}] = \Psi\Theta_{1,2}^{+} - Z_1\Theta_{1,2}^{\perp}.$$

3) Using the Matlab/Simulink software, two S-functions are written, the first for system (1) and the second for observer (2). According to Table I, the matrices $T_{k_+}, K_{k_+}, \Pi_{k_+}$, and $N_{\alpha(k-1),\alpha(k)}$ are updated with $K_{k_+} = K_{1k_+} + \Pi_{k_+} N_{\alpha(k-1),\alpha(k)}$.

Simulation results show, through Fig. 1(a) and (b), a good state-estimation performance. The estimation of the state x_4 is not presented due to space limitation.

Remark 8: If a common quadratic Lyapunov function $V(e_k, k) = e_k^T P e_k$ is imposed (i.e., $P_1 = P_2 = P$ and $U_1 = U_2 = U$), the corresponding LMIs are found to be unfeasible. Indeed, the polyquadratic stability is less conservative than the quadratic stability.

Remark 9: If the convex optimization, defined by (22), is applied, the following maximal bound γ is obtained for different value of h_{31_1}

h_{31_1}	1	1.2	1.26	1.27	1.28
$\gamma_{\max}$	184.3	98.87	49.53	35.05	1.1768

where the parameter h_{31_1} is a coefficient of the matrix $H_{\alpha(k)}$ of system (1). If h_{31_1} increases, then $\gamma_{\max}$ decreases, since there is a linear dependant of the nonlinear term $\phi(x_k, u_k, k)$.

V. Conclusion

A rigorous method for the design of observers for switched nonlinear descriptor systems in the presence of a UI has been presented. Existence conditions of such observers have been given and proved with a strict LMI formulation. Furthermore, a polyquadratic stability is used to assess the state estimation. It is interesting to note that the systems addressed in this paper are of a more general class than those reported in the literature. Moreover, from [27], an extension to design a robust observer for an uncertain switched descriptor system can be developed, and this is actually studied.

Appendix

It is proved that assumption A3 or (21) is equivalent to the existence of matrices Z_i such that $\Pi_{i,i}$ are Hurwitz.

Proof $A3 \Leftrightarrow (21)$: Define the following nonsingular matrices W_{1i}, W_3, and the full-column rank matrix W_{2i}

$$W_{1i} = \begin{bmatrix} I_n & 0 \\ -\Theta_{i,i}^{+}\varphi_{1i} & I_{2(n+q)} \end{bmatrix}, \quad W_{2i} = \begin{bmatrix} I_n & -\Psi\Theta_{i,i}^{+} \\ 0 & \Theta_{i,i}^{\perp} \\ 0 & \Theta_{i,i}\Theta_{i,i}^{+} \end{bmatrix}$$

$$W_3 = \begin{bmatrix} -I_n & 0 & 0 & 0 & 0 \\ zI_n & I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & -I_q & 0 \\ 0 & 0 & 0 & zI_q & I_q \end{bmatrix}.$$

According to Remark 5, for $\alpha(k+1) = \alpha(k) = i$ the existence of a solution $P_i > 0$, U_i of the LMI (17) needs that the matrix $\Pi_{i,i}$ is Hurwitz; therefore, each pair $(\Psi\Theta_{i,i}^{+}\varphi_{1i}, \quad \Theta_{i,i}^{\perp}\varphi_{1i})$ must be detectable. The proof is decomposed in two parts.

1) Let us prove that A3 is equivalent to

$$\text{rank}\begin{bmatrix} zI_n & \Psi \\ \varphi_{1i} & \Theta_{i,i} \end{bmatrix} - 2n - \text{rank}\, G_i = n + q \quad \forall\, |z| \geq 1, \quad i \in \varepsilon. \tag{23}$$

2) Let prove that (23) is equivalent to (21).

Proof 1: From W_3, the relation (23) is equivalent to

$$\text{rank}\begin{bmatrix} zI_n & \Psi \\ \varphi_{1i} & \Theta_i \end{bmatrix} W_3 - 2n - \text{rank}\, G_i = n + q \quad \forall\, |z| \geq 1, \quad i \in \varepsilon$$

which is equivalent to

$$\text{rank}\begin{bmatrix} zE_i - A_i & -F_i & 0 \\ zC_i & zG_i & G_i \\ C_i & G_i & 0 \end{bmatrix} - \text{rank}\, G_i = n + q \quad \forall\, |z| \geq 1, \quad i \in \varepsilon$$

which is equivalent to $A3$.

Proof 2: Since $\Theta_{i,i}^{+}\Theta_{i,i}\Theta_{i,i}^{+} = \Theta_{i,i}^{+}$, $\Theta_{i,i}\Theta_{i,i}^{+}\Theta_{i,i} = \Theta_{i,i}$, and

$$\text{rank}\begin{bmatrix} \Theta_{i,i} \\ \Psi \end{bmatrix} = \text{rank}\,\Theta_{i,i}$$

we obtain (23)

$$\Leftrightarrow \text{rank}\, W_{2i}\begin{bmatrix} zI_n & \Psi \\ \varphi_{1i} & \Theta_{i,i} \end{bmatrix} W_{1i} - 2n - \text{rank}\, G_i$$
$$= n + q \quad \forall\, |z| \geq 1, \quad i \in \varepsilon$$
$$\Leftrightarrow \text{rank}\,\Theta_{i,i} + \text{rank}\begin{bmatrix} zI_{n+m} - \Psi\Theta_{i,i}^{+}\varphi_{1i} \\ \Theta_{i,i}^{\perp}\varphi_{1i} \end{bmatrix} - 2n - \text{rank}\, G_i$$
$$= n + q \quad \forall\, |z| \geq 1, \quad i \in \varepsilon$$
$$\Leftrightarrow \text{rank}\begin{bmatrix} F_i \\ G_i \end{bmatrix} + \text{rank}\begin{bmatrix} zI_{n+m} - \Psi\Theta_{i,i}^{+}\varphi_{1i} \\ \Theta_{i,i}^{\perp}\varphi_{1i} \end{bmatrix}$$
$$= n + q \quad \forall\, |z| \geq 1, \quad i \in \varepsilon$$
$$\Leftrightarrow (22)$$

where $\text{rank}\,\Theta_{i,i} = 2n + \text{rank}\, G_i + \text{rank}\left[\begin{smallmatrix} F_i \\ G_i \end{smallmatrix}\right]$ and $\text{rank}\left[\begin{smallmatrix} F_i \\ G_i \end{smallmatrix}\right] = q$.

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An Extension of the Argument Principle and Nyquist Criterion to a Class of Systems With Unbounded Generators

Makan Fardad and Bassam Bamieh

***Abstract*—The Nyquist stability criterion is generalized to systems where the open-loop system has infinite-dimensional input and output spaces and an unbounded infinitesimal generator. The infinitesimal generator is assumed to be a sectorial operator with trace-class resolvent. The main result is obtained through use of the perturbation determinant and an extension of the argument principle to infinitesimal generators with trace-class resolvents.**

***Index Terms*—Argument principle, infinite-dimensional system, Nyquist stability criterion, perturbation determinant, unbounded infinitesimal generator.**

I. Introduction

The Nyquist criterion is of particular interest in system analysis as it offers a simple visual test to determine the stability of a closed-loop system for a family of feedback gains [1], [2]. Extensions of the Nyquist stability criterion exist for certain classes of distributed [3] and time-periodic [4] systems. Desoer and Wang [3] consider distributed systems in which the open-loop transfer function $G(s)$ belongs to the algebra of matrix-valued meromorphic functions of *finite* Euclidean dimension, and the Nyquist analysis is carried out by performing a coprime factorization on $G(s)$.

To motivate the discussion in this paper, let us first consider a finite-dimensional (multiinput multioutput) LTI system $G(s)$ placed in feedback with a constant gain γI. In analyzing the closed-loop stability of such a system, we are concerned with the eigenvalues in $\mathbb{C}^+$ of the closed-loop dynamics $A^{\rm cl}$. If s is an eigenvalue of $A^{\rm cl}$, then it satisfies $\det[sI - A^{\rm cl}] = 0$. Now to check whether the equation $\det[sI - A^{\rm cl}] = 0$ has solutions inside $\mathbb{C}^+$, one can apply the argument principle to $\det[I + \gamma G(s)]$ as s traverses some path $\mathfrak{D}$ enclosing $\mathbb{C}^+$. To elaborate, let us assume that we are given a state-space realization of the open-loop system. Then, using

$$\det[I + \gamma G(s)] = \frac{\det[sI - A^{\rm cl}]}{\det[sI - A]} \tag{1}$$

if one knows the number of unstable open-loop poles, one can determine the number of unstable closed-loop poles by looking at the plot of $\det[I + \gamma G(s)]\big|_{s\in\mathfrak{D}}$. But in the case of distributed systems, the open-loop and closed-loop infinitesimal generators $\mathcal{A}$ and $\mathcal{A}^{\rm cl}$ are operators on an infinite-dimensional Hilbert space $\mathcal{X}$ and can be *unbounded*. Hence, it is not clear how to define the characteristic functions $\det[s\mathcal{I} - \mathcal{A}]$ and $\det[s\mathcal{I} - \mathcal{A}^{\rm cl}]$. In this paper, we find an analog of (1) applicable to unbounded $\mathcal{A}$ and $\mathcal{A}^{\rm cl}$ and use operator-theoretic arguments to relate

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