

Design of observers for Takagi–Sugeno descriptor systems with unknown inputs and application to fault diagnosis

B. Marx, D. Koenig and J. Ragot

Abstract: A method for state-estimation of Takagi–Sugeno descriptor systems (TSDS) affected by unknown inputs (UI) has been presented here. For ease of implementation's sake, the proposed observers are not in descriptor form, but in usual form. Sufficient existence conditions of the unknown input observers (UIOs) are given and strict linear matrix inequalities are solved to determine the gain of the observers. If the perfect UI decoupling is not possible, the UIO is designed in order to minimise the \mathcal{L}_2 -gain from the UI to the state estimation error. The two previous objectives can be mixed in order to decouple the estimation to a subset of the UI, while attenuating the \mathcal{L}_2 -gain from the other UI to the estimation. The proposed UI observers are used for robust fault diagnosis. Fault diagnosis for TSDS is performed by designing a bank of observers. A simple decision logic and thresholds setting allow to determine the occurring fault. The results are established for both the continuous and the discrete time cases. The proposed method is illustrated by a numerical example.

1 Introduction

The Takagi–Sugeno (TS) model proposed by Takagi and Sugeno [1] is a well-known structure to represent nonlinear systems into several linear fuzzy models. In the last two decades, the control and the observation of TS systems have become challenging problems that received a considerable amount of attention. In [2], stability analysis and controller design are addressed, solutions are derived in the linear matrix inequality (LMI) formalism. Relaxed sufficient conditions for fuzzy controllers and fuzzy observers are proposed in [3, 4] via a multiple Lyapunov function approach.

The descriptor formalism is very attractive for system modelling, as pointed out in [5], since it describes a wider class of systems including physical systems with non-dynamic constraints (e.g. algebraic relations induced in interconnected systems such as power transfer networks or water distribution networks) or jump behaviour. The enhancement of the modelling ability is because of the structure of the dynamic equation, which encompasses not only dynamic equations, but also algebraic relations.

Since both TS and descriptor formalisms are attractive in the field of modelling, thus the TS representation has been generalised to descriptor systems. The stability and the design of state-feedback controllers for TS descriptor systems (TSDS) are characterised via LMI in [6, 7], in particular, the problem of nonlinear model following is treated in [7].

Robust output feedback, and H_∞ -control are considered for TSDS in [8] and [9], respectively. The study of TSDS is envisaged with interval methods in [10], in order to account the different operating points. Unfortunately, the problem of observer design, and especially the design of unknown input observers (UIOs), has resulted in very few works.

The design of UIO is a crucial problem since, in many practical cases, all input signals cannot be known. Moreover, this class of observers is widely used in the area of fault diagnosis, even if all the inputs are known (see Chapter 3 in [11]). The design of UIO has received considerable attention in the case of usual (in opposition to descriptor) linear systems [12], descriptor systems [13–15] or TS systems [16]. Unfortunately, to the best of the authors' knowledge, the design of UIO has not been treated in the generic case of TSDS. The aim of this paper is not only to generalise the existing works on UIO design to TSDS, but also to apply this new observer in the field of fault diagnosis of TSDS, which has not been treated so far.

This paper gives a simple extension to TSDS of the design of observers for the state estimation in the presence of UI. Under some sufficient conditions, the design of the observer is reduced to the determination of a matrix. The choice of this parameter is performed by solving strict LMIs. If the estimation error cannot be decoupled from the UI, an \mathcal{L}_2 observer is proposed to minimise the influence of the UI on the state estimation. The two design objectives can be mixed by decoupling the state estimation from a subset of the UI, and minimising the \mathcal{L}_2 -gain between the other UI and the state estimation error. The designed observers are used for fault diagnosis, since the UI can encompass the faults and the disturbances affecting the system. Designing several observers attenuating the disturbance effect, and decoupling the estimation from all faults, but one lead to the well-known generalised observer scheme (GOS) for fault diagnosis [11]. The design of observers is detailed both in the continuous time case and in the discrete time case.

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The paper is organised as follows: the class of studied systems is defined in Section 2 and the main results about UIO design are detailed in Section 3. Firstly, the definition of the UIO and the sufficient existence condition are established. Secondly, the computation of the gains of the observer is established. The design of \mathcal{L}_2 observers is treated in Section 4. Section 5 deals with the design of observers for both UI decoupling and disturbance attenuation. The application to fault diagnosis is studied in Section 6. Section 7 is devoted to a numerical example.

2 Takagi–Sugeno descriptor systems

To begin with, the class of systems considered in this paper is described. In the continuous time case, a TSDS is defined by

$$\mathbf{E}\dot{\mathbf{x}}(t) = \sum_{i=1}^r h_i(\mathbf{w}(t))(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t) + \mathbf{D}_i \mathbf{d}(t)) \quad (1)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{G}\mathbf{d}(t) \quad (2)$$

In the discrete time case, a TSDS is defined by

$$\mathbf{E}\mathbf{x}_{k+1} = \sum_{i=1}^r h_i(\mathbf{w}_k)(\mathbf{A}_i \mathbf{x}_k + \mathbf{B}_i \mathbf{u}_k + \mathbf{D}_i \mathbf{d}_k) \quad (3)$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{G}\mathbf{d}_k \quad (4)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state variable, $\mathbf{u} \in \mathbb{R}^{n_u}$ is the control input, $\mathbf{d} \in \mathbb{R}^g$ is the unknown input (disturbance, actuator noise or hidden message in the recovering framework) and $\mathbf{y} \in \mathbb{R}^m$ is the measured output. The matrices \mathbf{E} , \mathbf{A}_i , \mathbf{B}_i , \mathbf{D}_i , \mathbf{C} and \mathbf{G} are supposed to be real, known, constant and with appropriate dimensions according to the definition of the signals. The matrix \mathbf{E} may be singular. The activating functions, denoted $h_i(\mathbf{w}(t))$, for $i = 1, \dots, r$, are normalised, and satisfy the following constraints

$$0 \leq h_i(\mathbf{w}(t)) \leq 1, \quad \sum_{i=1}^r h_i(\mathbf{w}(t)) = 1, \quad \forall t$$

$$0 \leq h_i(\mathbf{w}_k) \leq 1, \quad \sum_{i=1}^r h_i(\mathbf{w}_k) = 1, \quad \forall k$$

The decision variable $\mathbf{w}(t)$ (or \mathbf{w}_k) is supposed to be real-time accessible, depending on the control input or on the measured output. Assume that the same matrix \mathbf{E} appears in all the different sub-models is not restrictive if we consider that the structure of the differential or algebraic relations is imposed by the physical structure of the system, which generally does not change with time. This formalism still encompasses the varying parameters or the nonlinearities since the matrices \mathbf{A}_i are different one from another, and since the activating functions introduce the nonlinear dynamics. An analogous argument justifies the single nature of the output matrix \mathbf{C} . The available measurements are determined by the location and the nature of the sensors, which generally do not change (the sensors are not removed during the operating time).

3 Design of UI decoupling observers

In this section, our aim is to design a multiple UIO. The UIO is used widely in the field of fault detection and isolation for dynamic systems, because the fault signals are generally unknown. Moreover, a measured signal can be considered

as unknown in order to isolate the default corrupting this particular signal (see Chapter 3 in [17]).

In this study, the proposed observers are not in descriptor form, in order to reduce the implementation complexity. In the continuous-time case, the proposed multiple UIO is defined by

$$\dot{\mathbf{z}}(t) = \sum_{i=1}^r h_i(\mathbf{w}(t))(\mathbf{N}_i \mathbf{z}(t) + \mathbf{M}_i \mathbf{u}(t) + \mathbf{L}_i \mathbf{y}(t)) \quad (5)$$

$$\hat{\mathbf{x}}(t) = \mathbf{z}(t) + \mathbf{T}_2 \mathbf{y}(t) \quad (6)$$

In the discrete-time case, the proposed multiple UIO is defined by

$$\mathbf{z}_{k+1} = \sum_{i=1}^r h_i(\mathbf{w}_k)(\mathbf{N}_i \mathbf{z}_k + \mathbf{M}_i \mathbf{u}_k + \mathbf{L}_i \mathbf{y}_k) \quad (7)$$

$$\hat{\mathbf{x}}_k = \mathbf{z}_k + \mathbf{T}_2 \mathbf{y}_k \quad (8)$$

The problem of unknown input decoupling observer (UIDO) design is to find the gains of the UIDO (5 and 6) [resp. (7 and 8)], namely \mathbf{N}_i , \mathbf{M}_i , \mathbf{L}_i and \mathbf{T}_2 , in order that the estimated state $\hat{\mathbf{x}}$ asymptotically tends to the state of (1 and 2) [resp. (3 and 4)]. In other words, the objective is that the estimation error defined by $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ [resp. $\mathbf{e}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$] tends to zero when $t \rightarrow \infty$ (resp. when $k \rightarrow \infty$), regardless of the unknown input the control input and the initial state.

Firstly, a sufficient rank condition for UI decoupling is given in Lemma 1. Secondly, a sufficient LMI condition for the convergence of the continuous-time UIO is given in Lemma 2 (it is extended to the discrete-time case in Corollary 1). Finally, the results are gathered in Theorem 1 and a design algorithm is given.

Lemma 1: There exists a continuous-time (resp. discrete-time) UIDO (7 and 8) for (1 and 2) [resp. (7 and 8) for (3 and 4)] if the following condition holds

$$\text{rank } \mathbf{X} = \text{rank} \begin{bmatrix} \mathbf{D}_1 & \cdots & \mathbf{D}_r \\ \mathbf{I}_r \otimes \mathbf{G} \end{bmatrix} + n + \text{rank } \mathbf{G} \quad (9)$$

Proof: The estimation error $\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is given by

$$\begin{aligned} \mathbf{e}(t) &= \mathbf{x}(t) - \hat{\mathbf{x}}(t) \\ &= \mathbf{x}(t) - \mathbf{z}(t) - \mathbf{T}_2 \mathbf{C}\mathbf{x}(t) - \mathbf{T}_2 \mathbf{G}\mathbf{d}(t) \end{aligned}$$

Assume that there exist \mathbf{T}_1 and \mathbf{T}_2 such that, the following equations hold

$$\mathbf{T}_1 \mathbf{E} + \mathbf{T}_2 \mathbf{C} = \mathbf{I}_n \quad (10)$$

$$\mathbf{T}_2 \mathbf{G} = \mathbf{0} \quad (11)$$

With (10) and (11), the estimation error becomes $\mathbf{e}(t) = \mathbf{T}_1 \mathbf{E}\mathbf{x}(t) - \mathbf{z}(t)$. Its time derivative is given by

$$\begin{aligned} \dot{\mathbf{e}}(t) &= \mathbf{T}_1 \mathbf{E}\dot{\mathbf{x}}(t) - \dot{\mathbf{z}}(t) \\ &= \sum_{i=1}^r h_i(\mathbf{w}(t))[\mathbf{T}_1(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t) + \mathbf{D}_i \mathbf{d}(t)) \\ &\quad - \mathbf{N}_i \mathbf{z}(t) - \mathbf{M}_i \mathbf{u}(t) - \mathbf{L}_i \mathbf{y}(t)] \\ &= \sum_{i=1}^r h_i(\mathbf{w}(t))[\mathbf{N}_i \mathbf{e}(t) \\ &\quad + (\mathbf{T}_1 \mathbf{A}_i - \mathbf{N}_i \mathbf{T}_1 \mathbf{E} - \mathbf{L}_i \mathbf{C})\mathbf{x}(t) \\ &\quad + (\mathbf{T}_1 \mathbf{B}_i - \mathbf{M}_i \mathbf{u}(t) + (\mathbf{T}_1 \mathbf{D}_i - \mathbf{L}_i \mathbf{G})\mathbf{d}(t)] \quad (12) \end{aligned}$$

The time derivative of the estimation error is given by

$$\dot{e}(t) = \sum_{i=1}^r h_i(w(t)) N_i e(t) \quad (13)$$

if the following constraints hold for $i = 1, \dots, r$

$$I_n = T_1 E + T_2 C \quad (14)$$

$$0 = T_2 G \quad (15)$$

$$0 = T_1 A_i - N_i T_1 E - L_i C \quad (16)$$

$$0 = T_1 B_i - M_i \quad (17)$$

$$0 = T_1 D_i - L_i G \quad (18)$$

In order to find the gains of the UIDO, according to the constraints (14–18), new parameters $K_i = N_i T_2 - L_i$ are introduced in (16). Then, the UIDO exists if, for $i = 1, \dots, r$, the following statements are true

$$N_i = T_1 A_i + K_i C \quad (19)$$

$$I_n = T_1 E + T_2 C \quad (20)$$

$$0 = T_2 G \quad (21)$$

$$0 = T_1 D_i + K_i G \quad (22)$$

$$M_i = T_1 B_i \quad (23)$$

$$L_i = N_i T_2 - K_i \quad (24)$$

Verifying the constraints (19–22) reduces to finding $\Theta \in \mathbb{R}^{n \times (n+m(r+1))}$ such that

$$\Theta X = Y \quad (25)$$

$$N_i = \Theta Y_i \quad (26)$$

where Θ , is given by

$$\Theta = [T_1 \quad T_2 | K_1 \quad K_2 \quad \dots \quad K_r] \quad (27)$$

Once Θ is known, M_i and L_i are deduced from (23) and (34), respectively. Equation (25) is solvable in the variable Θ if the following condition holds

$$\text{rank} \begin{bmatrix} X \\ Y \end{bmatrix} = \text{rank } X \quad (28)$$

where the matrices $X \in \mathbb{R}^{(n+m(r+1)) \times (n+q(r+1))}$ and $Y \in \mathbb{R}^{n \times (n+q(r+1))}$ are defined by

$$X = \left[\begin{array}{cc|ccc} E & 0_{n \times q} & D_1 & \dots & D_r \\ C & G & 0_{m \times q} & \dots & 0_{m \times q} \\ \hline 0_{rm \times n} & 0_{rm \times q} & I_r & \otimes G & \end{array} \right] \quad (29)$$

$$Y = [I_n \quad 0_{n \times q} | 0_{n \times rq}] \quad (30)$$

Obviously, with (30) and (29), the condition (28) becomes

$$\text{rank} \begin{bmatrix} X \\ Y \end{bmatrix} = n + \text{rank } G + \text{rank} \begin{bmatrix} D_1 & \dots & D_r \\ I_r & \otimes G & \end{bmatrix}$$

$$\text{rank } X = \text{rank} \left[\begin{array}{cc|ccc} E & 0 & D_1 & \dots & D_r \\ C & G & 0 & \dots & 0 \\ \hline 0 & 0 & I_r & \otimes G & \end{array} \right]$$

Then (28) is equivalent to (9). In the discrete-time case, the proof is very similar, thus it is omitted. \square

Lemma 2: The estimation error of the UIO (5 and 6) for (1 and 2) tends to zero if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $Z \in \mathbb{R}^{n \times (n+m(r+1))}$

verifying the following LMI for $i = 1, \dots, r$

$$(YX^+ Y_i)^T P + P Y X^+ Y_i + (X^+ Y_i)^T Z^T + Z X^+ Y_i < 0 \quad (31)$$

where \otimes is the Kronecker product. The matrices X and Y are defined by (29) and (30), respectively, and $Y_i \in \mathbb{R}^{(n+m(r+1)) \times n}$ are defined by

$$Y_i = \begin{bmatrix} A_i \\ 0_{m \times n} \\ e_i \otimes C \end{bmatrix}$$

where $e_i \in \mathbb{R}^{r \times 1}$ is the column vector with all its components equal to 0, except the i th equal to 1.

Proof: Suppose that (9) is satisfied, then (25) is solvable and the solutions Θ are given by

$$\Theta = YX^+ + ZX^\perp \quad (32)$$

where $Z \in \mathbb{R}^{n \times (n+m(r+1))}$ is an arbitrary matrix.

With (26) and (32), the matrices N_i are defined by $N_i = YX^+ Y_i + ZX^\perp Y_i$. The state estimation error tends to zero if the polytopic system (13) is stable. A well-known stability condition for polytopic system (see [18]) is the existence of a symmetric positive definite matrix P verifying $N_i^T P + P N_i < 0$ for $i = 1, \dots, r$. Then the UIDO provides an estimate of the system state if there exists a matrix Z such that $P(YX^+ Y_i + ZX^\perp Y_i) + (YX^+ Y_i + ZX^\perp Y_i)^T P < 0$, for all $i = 1, \dots, r$. Setting $Z = PZ$ then (31) is obtained, which completes the proof. \square

This result is extended to the discrete-time case.

Corollary 1: The estimation error of the UIO (7 and 8) for (3 and 4) tends to zero if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $Z \in \mathbb{R}^{n \times (n+m(r+1))}$ verifying the following LMI for $i = 1, \dots, r$

$$\begin{bmatrix} \Phi_i & (X^+ Y_i)^T Z^T \\ Z(X^+ Y_i) & -P \end{bmatrix} < 0 \quad (33)$$

where Φ_i is defined by

$$\Phi_i = (X^+ Y_i)^T Z^T (YX^+ Y_i) + (YX^+ Y_i)^T Z(X^+ Y_i) + (YX^+ Y_i)^T P(YX^+ Y_i) - P$$

Proof: Proof is similar to the continuous-time case, apart from the condition for the stability of the state estimation error. In the discrete-time case, e_k tends to zeros if there exists a symmetric positive definite matrix P such that $N_i^T P N_i - P > 0$ for $i = 1, \dots, r$. With $PZ = Z$, the LMI (33) follows. \square

Theorem 1: There exists a continuous-time (resp. discrete-time) UIO (5 and 6) for (1 and 2) [resp (7 and 8) for (3 and 4)] if the condition (9) is satisfied and if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ and a matrix $Z \in \mathbb{R}^{n \times (n+m(r+1))}$ verifying (31) [resp. (33)], for $i = 1, \dots, r$.

Finally, the design of UIO for continuous-time (resp. discrete-time) TSDS is reduced to the following procedure.

Step 1. Verify the existence condition (9).

Step 2. Solve the LMI (31) [resp. (33)] in P and Z .

Step 3. Compute Z with $Z = P^{-1} Z$. For a given Z , Θ is deduced from (32), the matrices N_i , M_i and L_i are derived from (19), (23) and (24), respectively.

This result unifies the results obtained, on the one hand, in the field of the descriptor systems with UI [19–21] and, on the other hand, in the field of the TS systems with UI [16]. It is useful because a TSDS cannot be reduced, either to a single singular system (it would not handle the nonlinearities because of the weighting functions h_i) nor to a regular TS system (it would not handle the algebraic relation between the state variables). The existence condition (9) can be linked to previous works concerning single descriptor systems [20, 21]. Considering (9) for a single descriptor system, would lead to the condition

$$\text{rank} \begin{bmatrix} \mathbf{E} & \mathbf{D}_1 & \mathbf{0} \\ \mathbf{C} & \mathbf{0} & \mathbf{G} \\ \mathbf{0} & \mathbf{G} & \mathbf{0} \end{bmatrix} = \text{rank} \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{G} \end{bmatrix} + n + \text{rank} \mathbf{G} \quad (34)$$

One can note that (34) is equivalent to the condition (21) or (31) of [20], and is also equivalent to the condition (A3a) in [21]. Moreover, the present paper gives only sufficient conditions, whereas [20] gave necessary and sufficient conditions. This difference appears because the present paper is basically written for TS systems, thus the weighting functions cause conservatism since the matrix $\sum_{i=1}^r h_i(\mathbf{w}(t))\mathbf{A}_i$ can take all the possible values in the polytope defined by its vertices \mathbf{A}_i .

4 Design of UIAOs

In this section, the aim is to design an observer for TSDS in order to minimise the influence of the UI on the state estimation when the perfect decoupling is not possible. The chosen criterion to minimise is the \mathcal{L}_2 -gain between the unknown input and the state estimation error. This approach is less restrictive than the design of a UIO since the structural condition (9) is partially relaxed.

As pointed out in the section of UIO design, the estimation error e is governed by a non-singular TS system (12), thus in order to bound the \mathcal{L}_2 -gain from the UI to e , and establish the sufficient conditions of the so-called \mathcal{L}_2 -observer, the following lemma concerning \mathcal{L}_2 -gain of TS-systems is needed.

Lemma 3 [18]: Consider the continuous-time TS-system defined by

$$\dot{\mathbf{x}}(t) = \sum_{i=1}^r h_i(\mathbf{w}(t))(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t)) \quad (35)$$

$$\mathbf{y}(t) = \sum_{i=1}^r h_i(\mathbf{w}(t))\mathbf{C}_i \mathbf{x}(t) \quad (36)$$

and the discrete-time TS-system defined by

$$\mathbf{x}_{k+1} = \sum_{i=1}^r h_i(\mathbf{w}_k)(\mathbf{A}_i \mathbf{x}_k + \mathbf{B}_i \mathbf{u}_k) \quad (37)$$

$$\mathbf{y}_k = \sum_{i=1}^r h_i(\mathbf{w}_k)\mathbf{C}_i \mathbf{x}_k \quad (38)$$

The system (35 and 36) [resp. (37 and 38)] is stable and verifies $\|\mathbf{y}\|_2 < \gamma \|\mathbf{u}\|_2$ if there exists a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ such that (39) [resp. (40)] is satisfied for $i = 1, \dots, r$.

$$\begin{bmatrix} \mathbf{A}_i^T \mathbf{P} + \mathbf{P} \mathbf{A}_i + \mathbf{C}_i^T \mathbf{C}_i & \mathbf{P} \mathbf{B}_i \\ \mathbf{B}_i^T \mathbf{P} & -\gamma^2 \mathbf{I} \end{bmatrix} < 0 \quad (39)$$

$$\begin{bmatrix} \mathbf{A}_i^T \mathbf{P} \mathbf{A}_i + \mathbf{C}_i^T \mathbf{C}_i - \mathbf{P} & \mathbf{A}_i^T \mathbf{P} \mathbf{B}_i \\ \mathbf{B}_i^T \mathbf{P} \mathbf{A}_i & \mathbf{B}_i^T \mathbf{P} \mathbf{B}_i - \gamma^2 \mathbf{I} \end{bmatrix} < 0 \quad (40)$$

For a given real-positive γ , an observer is said to be an unknown input attenuating observer (UIAO) of \mathcal{L}_2 -gain γ , if the state estimation error, e , and the unknown input, d , satisfy $\|e\|_2 < \gamma \|d\|_2$.

Theorem 2: There exists a UIAO (51 and 61), with an \mathcal{L}_2 -gain lower than γ , for the system (1 and 2), if the condition (41) is satisfied, and if there exist a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ and matrices $\mathbf{Z} \in \mathbb{R}^{n \times (n+m)}$ and $\mathbf{K}_i \in \mathbb{R}^{n \times m}$, verifying the LMI (42) for $i = 1, \dots, r$.

$$\text{rank} \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{C} & \mathbf{G} \end{bmatrix} = n + \text{rank} \mathbf{G} \quad (41)$$

$$\begin{bmatrix} \Psi_{i,1} & \Psi_{i,2} \\ \Psi_{i,2}^T & -\gamma^2 \mathbf{I}_q \end{bmatrix} < 0 \quad (42)$$

where the matrices $\Psi_{i,1}$, and $\Psi_{i,2}$ are given by

$$\begin{aligned} \Psi_{i,1} &= \mathbf{P} \mathbf{Y} \mathbf{X}_1^+ \mathbf{A}_i + \mathbf{Z} \mathbf{X}_1^+ \mathbf{A}_i + \mathbf{K}_i \mathbf{C} \\ &\quad + (\mathbf{P} \mathbf{Y} \mathbf{X}_1^+ \mathbf{A}_i + \mathbf{Z} \mathbf{X}_1^+ \mathbf{A}_i + \mathbf{K}_i \mathbf{C})^T + \mathbf{I}_n \end{aligned}$$

$$\Psi_{i,2} = \mathbf{P} \mathbf{Y} \mathbf{X}_1^+ \mathbf{D}_i + \mathbf{Z} \mathbf{X}_1^+ \mathbf{D}_i + \mathbf{K}_i \mathbf{G}$$

where $\mathbf{X}_1^+ \in \mathbb{R}^{(n+q) \times n}$, $\mathbf{X}_2^+ \in \mathbb{R}^{(n+q) \times m}$, $\mathbf{X}_1^\perp \in \mathbb{R}^{(n+m) \times n}$ and $\mathbf{X}_2^\perp \in \mathbb{R}^{(n+m) \times m}$ are defined by

$$\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{C} & \mathbf{G} \end{bmatrix}^+ = [\mathbf{X}_1^+ \quad \mathbf{X}_2^+]$$

$$\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{C} & \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{C} & \mathbf{G} \end{bmatrix}^+ - \mathbf{I}_{n+m} = [\mathbf{X}_1^\perp \quad \mathbf{X}_2^\perp]$$

Proof: If (41) is satisfied, then there exist \mathbf{T}_1 and \mathbf{T}_2 such that

$$[\mathbf{T}_1 \quad \mathbf{T}_2] \mathbf{X} = \mathbf{Y}$$

where \mathbf{X} and \mathbf{Y} are given by

$$\mathbf{X} = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{C} & \mathbf{G} \end{bmatrix}, \quad \mathbf{Y} = [\mathbf{I}_n \quad \mathbf{0}_{n \times m}]$$

and, for any arbitrary matrix \mathbf{Z} , \mathbf{T}_1 and \mathbf{T}_2 are given by

$$\mathbf{T}_1 = \mathbf{Y} \mathbf{X}_1^+ + \mathbf{Z} \mathbf{X}_1^\perp \quad (43)$$

$$\mathbf{T}_2 = \mathbf{Y} \mathbf{X}_2^+ + \mathbf{Z} \mathbf{X}_2^\perp \quad (44)$$

Following the proof of Theorem 1, if (19), (23) and (24) hold, the state estimation error $e(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ is governed by

$$\dot{e}(t) = \sum_{i=1}^r h_i(\mathbf{w}(t))((\mathbf{T}_1 \mathbf{A}_i + \mathbf{K}_i \mathbf{C})e(t) + (\mathbf{T}_1 \mathbf{D}_i + \mathbf{K}_i \mathbf{G})d(t)) \quad (45)$$

According to Lemma 3, $\|e(t)\|_2 < \gamma \|d(t)\|_2$ if there exists a symmetric positive definite matrix \mathbf{P} such that the following LMI hold for $i = 1, \dots, r$

$$\begin{bmatrix} \Psi_i + \mathbf{I}_n & \mathbf{P} \mathbf{T}_1 \mathbf{D}_i + \mathbf{P} \mathbf{K}_i \mathbf{G} \\ \mathbf{D}_i^T \mathbf{T}_1^T \mathbf{P} + \mathbf{G}^T \mathbf{K}_i^T \mathbf{P} & -\gamma^2 \mathbf{I}_q \end{bmatrix} < 0$$

where Ψ_i is given by

$$\Psi_i = (\mathbf{T}_1 \mathbf{A}_i + \mathbf{K}_i \mathbf{C})^T \mathbf{P} + \mathbf{P}(\mathbf{T}_1 \mathbf{A}_i + \mathbf{K}_i \mathbf{C})$$

With $\underline{\mathbf{K}}_i = \mathbf{P} \mathbf{K}_i$ and $\underline{\mathbf{Z}} = \mathbf{P} \mathbf{Z}$, the LMI (42) follows, which completes the proof. \square

Remark 1: Obviously, condition (41) is less restrictive than (9).

Remark 2: γ_2 can be considered as a variable to be minimised during the LMI optimisation, to obtain an optimal UI attenuation

Corollary 2: There exists a UIAO (7 and 8) with an \mathcal{L}_2 -gain lower than a given real positive γ for the system (3 and 4), if the condition (41) is satisfied, and if there exist a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$, and matrices $\underline{\mathbf{Z}} \in \mathbb{R}^{n \times (n+m)}$ and $\underline{\mathbf{K}}_i \in \mathbb{R}^{n \times m}$, verifying the following LMI for $i = 1, \dots, r$

$$\begin{bmatrix} \mathbf{I}_n - \mathbf{P} & \mathbf{0} & \Phi_{i,1}^T \\ \mathbf{0} & -\gamma^2 \mathbf{I}_n & \Phi_{i,2}^T \\ \Phi_{i,1} & \Phi_{i,2} & -\mathbf{P} \end{bmatrix} < 0 \quad (46)$$

where $\Phi_{i,1}$ and $\Phi_{i,2}$ are defined by

$$\Phi_{i,1} = \mathbf{P} \mathbf{Y} \mathbf{X}^+ \mathbf{A}_i + \underline{\mathbf{Z}} \mathbf{X}_1^+ \mathbf{A}_i + \underline{\mathbf{K}}_i \mathbf{C}$$

$$\Phi_{i,2} = \mathbf{P} \mathbf{Y} \mathbf{X}_1^+ \mathbf{D}_i + \underline{\mathbf{Z}} \mathbf{X}_1^+ \mathbf{D}_i + \underline{\mathbf{K}}_i \mathbf{G}$$

Proof: The proof follows the lines of the proof of Theorem 2 with a Schur complement and is therefore omitted. \square

Finally the design of UIAO for continuous-time (resp. discrete-time) TSDS is reduced to the following procedure.

Step 1. Verify the existence condition (41).

Step 2. Solve the LMI (42) [resp. (46)] in \mathbf{P} , $\underline{\mathbf{Z}}$ and $\underline{\mathbf{K}}_i$.

Step 3. Compute \mathbf{Z} and \mathbf{K}_i with $\mathbf{Z} = \mathbf{P}^{-1} \underline{\mathbf{Z}}$ and $\mathbf{K}_i = \mathbf{P}^{-1} \underline{\mathbf{K}}_i$, respectively. The matrices \mathbf{T}_1 and \mathbf{T}_2 are obtained by (43) and (44). The matrices \mathbf{N}_i , \mathbf{M}_i and \mathbf{L}_i are derived from (19), (23) and (24), respectively.

5 Design of unknown input decoupling and attenuating observer

If the UIs are too numerous or if their distribution structure makes the perfect UI decoupling of the estimation impossible [i.e. if the structural condition (9) is not satisfied] a compromise can be made in order to design an observer ensuring two complementary objectives with less restrictive existence conditions. Firstly, the state estimation is perfectly decoupled to a subset of the UI denoted $\mathbf{d}(t)$. Secondly, the \mathcal{L}_2 -gain between the other UIs, denoted $\bar{\mathbf{d}}(t)$, to the state estimation error is minimised, thus the state estimation is made maximally robust to these UI. Partitioning the UI into $\mathbf{d}(t)$ and $\bar{\mathbf{d}}(t)$ the system (1 and 2) can be written as

$$\begin{aligned} \mathbf{E} \dot{\mathbf{x}}(t) &= \sum_{i=1}^r h_i(\mathbf{w}(t)) (\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t) \\ &\quad + \mathbf{D}_i \mathbf{d}(t) + \bar{\mathbf{D}}_i \bar{\mathbf{d}}(t)) \end{aligned} \quad (47)$$

$$\mathbf{y}(t) = \mathbf{C} \mathbf{x}(t) + \mathbf{G} \mathbf{d}(t) + \bar{\mathbf{G}} \bar{\mathbf{d}}(t) \quad (48)$$

where $\mathbf{d}(t) \in \mathbb{R}^q$ and $\bar{\mathbf{d}}(t) \in \mathbb{R}^{\bar{q}}$. The partition of the UI into $\mathbf{d}(t)$ and $\bar{\mathbf{d}}(t)$ is such that the perfect decoupling condition is satisfied for $(\mathbf{E}, \mathbf{C}, \mathbf{G}, \mathbf{D}_1, \dots, \mathbf{D}_r)$, then the \mathcal{L}_2 -gain from $\bar{\mathbf{d}}(t)$ to the state estimation error is minimised.

The designs of UIDO and UIAO are combined to derive the design of a UI decoupling/attenuating observer (UIDAO). The sufficient existence conditions are given in the following theorem.

Theorem 3: There exists an observer (5 and 6) ensuring perfect decoupling to $\mathbf{d}(t)$ and maximally robust to $\bar{\mathbf{d}}(t)$ if condition (49) is satisfied and if there exist a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ and a matrix $\underline{\mathbf{Z}} \in \mathbb{R}^{n \times (n+(r+1)m)}$ solution of the minimisation of γ under the LMI constraint (50) for $i = 1, \dots, r$.

$$\text{rank } \bar{\mathbf{X}} = n + \text{rank} = [\mathbf{G} \quad \bar{\mathbf{G}}] + \text{rank} \begin{bmatrix} \mathbf{D}_1 & \cdots & \mathbf{D}_r \\ \mathbf{I}_r \otimes \mathbf{G} \end{bmatrix} \quad (49)$$

$$\begin{bmatrix} \bar{\Psi}_{i,1} & \bar{\Psi}_{i,2} \\ \bar{\Psi}_{i,1}^T & -\gamma^2 \mathbf{I} \end{bmatrix} < 0 \quad (50)$$

where $\bar{\Psi}_{i,1}$ and $\bar{\Psi}_{i,2}$ are given by

$$\begin{aligned} \bar{\Psi}_{i,1} &= \mathbf{P} \mathbf{Y} \mathbf{X}^+ \mathbf{Y}_i + \bar{\mathbf{Z}} \mathbf{X}^+ \mathbf{Y}_i \\ &\quad + (\bar{\mathbf{Y}} \mathbf{X}^+ \mathbf{Y}_i)^T \mathbf{P} + (\bar{\mathbf{X}}^+ \mathbf{Y}_i)^T \bar{\mathbf{Z}}^T + \mathbf{I}_n \end{aligned}$$

$$\bar{\Psi}_{i,2} = \mathbf{P} \mathbf{Y} \mathbf{X}^+ \bar{\mathbf{Y}}_i + \bar{\mathbf{Z}} \mathbf{X}^+ \bar{\mathbf{Y}}_i$$

where $\bar{\mathbf{X}}^+$ is the pseudo-inverse of $\bar{\mathbf{X}}$, $\bar{\mathbf{X}}^+ = \mathbf{I} - \bar{\mathbf{X}} \bar{\mathbf{X}}^+$, and where $\bar{\mathbf{X}}$, $\bar{\mathbf{Y}}$, $\bar{\mathbf{Y}}_i$ and $\bar{\mathbf{Y}}_i$ are given by

$$\bar{\mathbf{X}} = \begin{bmatrix} \mathbf{E} & \mathbf{0}_{n \times q} & \mathbf{0}_{n \times \bar{q}} & \mathbf{D}_1 & \cdots & \mathbf{D}_r \\ \mathbf{C} & \mathbf{G} & \bar{\mathbf{G}} & \mathbf{0}_{m \times q} & \cdots & \mathbf{0}_{m \times q} \\ \mathbf{0}_{rm \times n} & \mathbf{0}_{rm \times q} & \mathbf{0}_{rm \times \bar{q}} & \mathbf{I}_r \otimes \mathbf{G} & & \end{bmatrix}$$

$$\bar{\mathbf{Y}} = [\mathbf{I}_n \quad \mathbf{0}_{n \times q + \bar{q}} \quad \mathbf{0}_{n \times r q}]$$

$$\mathbf{Y}_i = \begin{bmatrix} \mathbf{A}_i \\ \mathbf{0}_{m \times n} \\ \mathbf{e}_i \otimes \mathbf{C} \end{bmatrix}, \quad \bar{\mathbf{Y}}_i = \begin{bmatrix} \bar{\mathbf{D}}_i \\ \mathbf{0}_{m \times \bar{q}} \\ \mathbf{e}_i \otimes \bar{\mathbf{G}} \end{bmatrix}$$

Proof: The state estimation error $\mathbf{e}(t)$ is governed by the following system

$$\begin{aligned} \dot{\mathbf{e}}(t) &= \sum_{i=1}^r h_i(\mathbf{w}(t)) (\mathbf{N}_i \mathbf{e}(t) + (\mathbf{T}_1 \mathbf{A}_i - \mathbf{N}_i \mathbf{T}_1 \mathbf{E} - \mathbf{L}_i \mathbf{C}) \mathbf{x}(t) \\ &\quad + (\mathbf{T}_1 \mathbf{B}_i - \mathbf{M}_i) \mathbf{u}(t) + (\mathbf{T}_1 \bar{\mathbf{D}}_i - \mathbf{L}_i \bar{\mathbf{G}}) \bar{\mathbf{d}}(t) \\ &\quad + (\mathbf{T}_1 \mathbf{D}_i - \mathbf{L}_i \mathbf{G}) \mathbf{d}(t) + \mathbf{T}_2 \mathbf{G} \dot{\mathbf{d}}(t) + \mathbf{T}_2 \bar{\mathbf{G}} \dot{\bar{\mathbf{d}}}(t) \end{aligned} \quad (52)$$

Following a similar argument to that in the proof of Theorem 1, if the observer parameters satisfy the constraints (19–24) and $\mathbf{T}_2 \bar{\mathbf{G}} = \mathbf{0}$, $\mathbf{e}(t)$ is governed by

$$\dot{\mathbf{e}}(t) = \sum_{i=1}^r h_i(\mathbf{w}(t)) (\mathbf{N}_i \mathbf{e}(t) + (\mathbf{T}_1 \bar{\mathbf{D}}_i + \mathbf{K}_i \bar{\mathbf{G}}) \bar{\mathbf{d}}(t)) \quad (53)$$

These constraints can be written as $\Theta \bar{\mathbf{X}} = \bar{\mathbf{Y}}$, with Θ defined by (27). This equation can be solved if and only if $\text{rank}[\bar{\mathbf{X}}^T \quad \bar{\mathbf{Y}}^T]^T = \text{rank } \bar{\mathbf{X}}$, which is equivalent to the condition (49). If condition (49) is satisfied, then, for any arbitrary matrix \mathbf{Z} , Θ is given by

$$\Theta = \bar{\mathbf{Y}} \bar{\mathbf{X}}^+ + \mathbf{Z} \bar{\mathbf{X}}^\perp \quad (54)$$

Then, the only parameter to be found is the matrix \mathbf{Z} . Since the matrices \mathbf{N}_i and $(\mathbf{T}_1 \bar{\mathbf{D}}_i - \mathbf{L}_i \bar{\mathbf{G}})$ can be written as

$$\mathbf{N}_i = \Theta \mathbf{Y}_i = \bar{\mathbf{Y}} \mathbf{X}^+ \mathbf{Y}_i + \mathbf{Z} \bar{\mathbf{X}}^\perp \mathbf{Y}_i$$

$$\mathbf{T}_1 \bar{\mathbf{D}}_i + \mathbf{K}_i \bar{\mathbf{G}} = \Theta \bar{\mathbf{Y}}_i = \bar{\mathbf{Y}} \mathbf{X}^+ \bar{\mathbf{Y}}_i + \bar{\mathbf{Z}} \mathbf{X}^\perp \bar{\mathbf{Y}}_i$$

the stability condition of (53) follows the same lines as in the proof of Theorem 2. Rewriting the stability condition (39) for the triplet $(\mathbf{N}_i, (\mathbf{T}_1 \bar{\mathbf{D}}_i - \mathbf{K}_i \bar{\mathbf{G}}), \mathbf{L}_n)$, and setting $\mathbf{P}\mathbf{Z} = \bar{\mathbf{Z}}$, the LMI condition (50) follows. \square

Remark 3: Obviously, condition (49) is less restrictive than (9). To obtain perfect decoupling to all the UI of (47 and 48), \mathbf{D}_i and \mathbf{G} should be replaced by $[\mathbf{D}_i \ \bar{\mathbf{D}}_i]$ and $[\mathbf{G} \ \bar{\mathbf{G}}]$, respectively, in (49) and (51), which would lead to a more restrictive existence condition.

A similar result can be given for discrete-time systems defined by

$$\mathbf{E}\mathbf{x}_{k+1} = \sum_{i=1}^r h_i(\mathbf{w}_k)(\mathbf{A}_i \mathbf{x}_k + \mathbf{B}_i \mathbf{u}_k + \mathbf{D}_i \mathbf{d}_k + \bar{\mathbf{D}}_i \bar{\mathbf{d}}_k) \quad (55)$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k + \mathbf{G}\mathbf{d}_k + \bar{\mathbf{G}}\bar{\mathbf{d}}_k \quad (56)$$

Corollary 3: There exists an observer (7 and 8) ensuring perfect decoupling to \mathbf{d}_k and maximally robust to $\bar{\mathbf{d}}_k$ if condition (49) is satisfied and if there exist a symmetric positive definite matrix $\mathbf{P} \in \mathbb{R}^{n \times n}$ and a matrix $\bar{\mathbf{Z}} \in \mathbb{R}^{n \times (n+(r+1)m)}$ solution to the minimisation of γ under the following LMI constraint for $i = 1, \dots, r$.

$$\begin{bmatrix} \mathbf{I}_n - \mathbf{P} & \mathbf{0} & \Phi_{i,1}^\top \\ \mathbf{0} & -\gamma^2 \mathbf{I} & \Phi_{i,2}^\top \\ \Phi_{i,1} & \Phi_{i,2} & -\mathbf{P} \end{bmatrix} < 0 \quad (57)$$

where $\bar{\Phi}_{i,1}$ and $\bar{\Phi}_{i,2}$ are given by

$$\begin{aligned} \bar{\Phi}_{i,1} &= \bar{\mathbf{P}} \bar{\mathbf{Y}} \mathbf{X}^+ \bar{\mathbf{Y}}_i + \bar{\mathbf{Z}} \mathbf{X}^\perp \bar{\mathbf{Y}}_i \\ \bar{\Phi}_{i,2} &= \bar{\mathbf{P}} \bar{\mathbf{Y}} \mathbf{X}^+ \bar{\mathbf{Y}}_i + \bar{\mathbf{Z}} \mathbf{X}^\perp \bar{\mathbf{Y}}_i \end{aligned}$$

Proof: The proof follows the lines of the proof of Corollary 3 and Theorem 3, and is therefore omitted. \square

Finally the design of the observer for continuous-time (resp. discrete-time) TSDS is reduced to the following procedure.

- Step 1.* Verify the existence condition (49).
- Step 2.* Solve the LMI (50) [resp. (57)] in \mathbf{P} and $\bar{\mathbf{Z}}$.
- Step 3.* Compute \mathbf{Z} with $\mathbf{Z} = \mathbf{P}^{-1} \bar{\mathbf{Z}}$. For a given \mathbf{Z} , Θ is given by (54), then the matrices \mathbf{T}_1 , \mathbf{T}_2 and \mathbf{K}_i are obtained. The matrices \mathbf{N}_i , \mathbf{M}_i and \mathbf{L}_i are derived from (19), (23) and (24), respectively.

6 Application to fault diagnosis

In this section, the UI decoupling and attenuating observers are used to perform fault diagnosis. Consider a continuous-time TSDS affected by faults $\mathbf{f}(t) \in \mathbb{R}^q$ and disturbances $\mathbf{w}(t) \in \mathbb{R}^{\bar{q}}$ defined by

$$\begin{aligned} \mathbf{E}\dot{\mathbf{x}}(t) &= \sum_{i=1}^r h_i(\mathbf{w}(t))(\mathbf{A}_i \mathbf{x}(t) + \mathbf{B}_i \mathbf{u}(t) \\ &\quad + \mathbf{D}_{fi} \mathbf{f}(t) + \mathbf{D}_{wi} \mathbf{w}(t)) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{G}_f \mathbf{f}(t) + \mathbf{G}_w \mathbf{w}(t) \end{aligned}$$

In the discrete-time case, the TSDS affected by faults

$\mathbf{f}_k \in \mathbb{R}^q$ and disturbances $\mathbf{w}_k \in \mathbb{R}^{\bar{q}}$ is defined by

$$\begin{aligned} \mathbf{E}\mathbf{x}_{k+1} &= \sum_{i=1}^r h_i(\mathbf{w}_k)(\mathbf{A}_i \mathbf{x}_k + \mathbf{B}_i \mathbf{u}_k + \mathbf{D}_{fi} \mathbf{f}_k + \mathbf{D}_{wi} \mathbf{w}_k) \\ \mathbf{y}_k &= \mathbf{C}\mathbf{x}_k + \mathbf{G}_f \mathbf{f}_k + \mathbf{G}_w \mathbf{w}_k \end{aligned}$$

It is assumed that each component of the disturbance vector is bounded, and that the value of this bound is known : $|w_i(t)| < v_i$ (resp $|w_{ik}| < v_i$ in the discrete-time case) for $i = 1, \dots, \bar{q}$ for all t (resp. for all k). The well-known GOS [11] can be applied to propose a method for the fault diagnosis of TSDS. In this approach, q UIDAO are designed. The ℓ th UIDAO is designed by considering the ℓ th fault as a UI. A subset, denoted Σ_ℓ , of the disturbances can also be considered as UI provided the existence condition (49) is satisfied. In other words, the ℓ th observer is designed for the system (47 and 48) [resp. (55 and 56)], with

$$\begin{aligned} \mathbf{D}_i &= [\mathbf{D}_{fi}^\ell | \mathbf{D}_{wi}^j, j \in \Sigma_\ell], & \bar{\mathbf{D}}_i &= [\mathbf{D}_{wi}^j, j \in \bar{\Sigma}_\ell] \\ \mathbf{G} &= [\mathbf{G}_f^\ell | \mathbf{G}_w^j, j \in \Sigma_\ell], & \bar{\mathbf{G}} &= [\mathbf{G}_w^j, j \in \bar{\Sigma}_\ell] \end{aligned}$$

where \mathbf{M}^ℓ denotes the ℓ th column of the matrix \mathbf{M} , and $\bar{\Sigma}_\ell$ denotes the complementary to Σ_ℓ in $\{1, 2, \dots, \bar{q}\}$.

As a consequence, the output estimation of the ℓ th observer will be sensitive to all the faults but the ℓ th, insensitive to the ℓ th fault and to a subset of the disturbances, denoted Σ_ℓ , and maximally robust to the other disturbances belonging to the subset $\bar{\Sigma}_\ell$. The subsets of the UI are determined so that the decoupling condition (49) is satisfied for all the disturbances in Σ_ℓ and so that the \mathcal{L}_2 -gain from the disturbances in $\bar{\Sigma}_\ell$ to the output estimation error is minimised. In other words, the output estimation error is a residual signal. A classical method for observer-based fault diagnosis is to suppose the occurrence of the ℓ th fault if all residual signals, except the ℓ th, are significantly different from zero. The problem is then to quantify the term significantly. In order to discriminate between the influence of the disturbances and the ℓ th fault, one can compute the \mathcal{L}_2 -gain from the disturbances to each output estimation error, as described in the following procedure.

- For each fault $\mathbf{f}_\ell(t)$ (resp. $\mathbf{f}_{\ell k}$ in the discrete-time case) Design the UIDAO, sensitive to all faults except $\mathbf{f}_\ell(t)$ (resp. $\mathbf{f}_{\ell k}$), insensitive to $\mathbf{w}_i(t)$ (resp. \mathbf{w}_{ik}) $i \in \Sigma_\ell$ and maximally robust to $\mathbf{w}_i(t)$ (resp. \mathbf{w}_{ik}), $i \in \bar{\Sigma}_\ell$.
- Compute the norm-bound of the attenuated disturbances, denoted ρ_ℓ

$$\rho_\ell = \sqrt{\sum_{i \in \bar{\Sigma}_\ell} v_i^2}$$

- For each component of the output $\mathbf{y}_j(t)$ (resp. \mathbf{y}_{jk}), compute the \mathcal{L}_2 -gain, denoted $g_{\ell j}$, from the attenuated disturbances $\{\mathbf{w}_i(t) | i \in \bar{\Sigma}_\ell\}$ (resp. $\{\mathbf{w}_{ik} | i \in \bar{\Sigma}_\ell\}$) to the j th output estimation error and compute the boolean vector $\mathbf{b}_\ell(t) = [b_{\ell 1}(t) \ b_{\ell 2}(t) \ \dots \ b_{\ell m}(t)]$ (resp. $\mathbf{b}_{\ell k} = [b_{\ell 1k} \ b_{\ell 2k} \ \dots \ b_{\ell mk}]$), where $b_{\ell j}(t)$ (resp. $b_{\ell jk}$) is defined by

$$\begin{aligned} b_{\ell j}(t) &= \begin{cases} 1, & \text{if } |\hat{\mathbf{y}}_{\ell j}(t) - \mathbf{y}_j(t)| > \alpha g_{\ell j} \rho_\ell \\ 0, & \text{else} \end{cases} \\ b_{\ell jk} &= \begin{cases} 1, & \text{if } |\hat{\mathbf{y}}_{\ell jk} - \mathbf{y}_{jk}| > \alpha g_{\ell j} \rho_\ell \\ 0, & \text{else} \end{cases} \end{aligned}$$

where $\hat{\mathbf{y}}_{\ell j}(t)$ (resp. $\hat{\mathbf{y}}_{\ell jk}$) is the j th component of the estimated

output given by the ℓ th observer. The positive scalar α allows the designer to handle the compromise between non-detection and false alarm (e.g. considering the accuracy of the model).

• Compute the alarm $a_\ell(t)$ (resp. $a_{\ell k}$), affected to $f_\ell(t)$ (resp. $f_{\ell k}$), defined in the continuous-time case by

$$a_\ell(t) = \begin{cases} 1, & \text{if } (\mathbf{b}_i(t)\mathbf{b}_i^\top(t) \geq 1, \forall i \neq \ell) \\ & \& (\mathbf{b}_\ell(t)\mathbf{b}_\ell^\top(t) = 0) \\ 0, & \text{else} \end{cases}$$

or in the discrete-time case by

$$a_{\ell k} = \begin{cases} 1, & \text{if } (\mathbf{b}_{ik}\mathbf{b}_{ik}^\top \geq 1, \forall i \neq \ell) \& (\mathbf{b}_{\ell k}\mathbf{b}_{\ell k}^\top = 0) \\ 0, & \text{else} \end{cases}$$

This approach can be conservative, since the only available information about the disturbances is their amplitude bound and the \mathcal{L}_2 -gain of their influence onto the output estimation error, thus it may imply non-detection. This effect can be limited by the use of the parameter α , which can be adjusted according to measurements of the system under healthy operation. Nevertheless, comparing each component of the estimation error with a threshold makes it possible to avoid distributing a significant error affecting a component on all the various components, and then reduces the non-detection.

One should note that the GOS is an efficient structure of diagnosis in order to detect and isolate single faults. In the case of simultaneous faults, two faults may cause non-zero residue responses in all observers. In this case, the dedicated observer scheme can be considered as an alternative, but the decoupling conditions become much more restrictive since all the fault inputs, but the ℓ th have to be decoupled from the ℓ th residue. This scheme is not detailed here, but can readily be applied since it suffices to change the definition of the matrices \mathbf{D}_i , \mathbf{D}_i^j , \mathbf{G} and \mathbf{G}^j : the ℓ th observer should be designed for the system (47 and 48) [resp. (55 and 56)], with

$$\mathbf{D}_i = \left[\mathbf{D}_{fi}^1 \quad \dots \quad \mathbf{D}_{fi}^{\ell-1} \quad \mathbf{D}_{fi}^{\ell+1} \quad \dots \quad \mathbf{D}_{fi}^q \mid \mathbf{D}_{wi}^j, j \in \Sigma_\ell \right]$$

$$\mathbf{G} = \left[\mathbf{G}_{fi}^1 \quad \dots \quad \mathbf{G}_{fi}^{\ell-1} \quad \mathbf{G}_{fi}^{\ell+1} \quad \dots \quad \mathbf{G}_{fi}^q \mid \mathbf{G}_w^j, j \in \Sigma_\ell \mathbf{e} \right]$$

$$\bar{\mathbf{D}}_i = \left[\mathbf{D}_{wi}^j, j \in \bar{\Sigma}_\ell \right], \quad \bar{\mathbf{G}} = \left[\mathbf{G}_w^j, j \in \bar{\Sigma}_\ell \right]$$

7 Design example

In this section, the proposed approach for fault diagnosis is illustrated. Consider a discrete-time TSDS defined by

$$\mathbf{E}\mathbf{x}_{k+1} = \sum_{i=1}^2 h_i(\mathbf{w}_k)(\mathbf{A}_i\mathbf{x}_k + \mathbf{B}_i\mathbf{u}_k + \mathbf{D}_{fi}\mathbf{f}_k + \mathbf{D}_{wi}\mathbf{w}_k)$$

$$\mathbf{y}_k = \mathbf{C}\mathbf{x}_k$$

with $\mathbf{E} = \text{diag } 1 \ 1 \ 1 \ 0$ and

$$\mathbf{A}_1 = \begin{pmatrix} -0.5 & -0.5 & 0.2 & 0.2 \\ -0.9 & 0.1 & 0.4 & 0.7 \\ -0.2 & -0.7 & 0 & 0 \\ -0.2 & -0.4 & 0.4 & 0 \end{pmatrix},$$

$$\mathbf{B}_1 = \mathbf{D}_{f1} = \begin{pmatrix} 5 & 0 \\ 0 & 0 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{A}_2 = \begin{pmatrix} 0.1 & -0.2 & 0.4 & 0.9 \\ -0.2 & 0.6 & -0.2 & -0.7 \\ 0.5 & -0.7 & -0.7 & 0.6 \\ -0.7 & 0.4 & 0.4 & 3.6 \end{pmatrix},$$

$$\mathbf{B}_2 = \mathbf{D}_{f2} = \begin{pmatrix} 6 & 0 \\ 0 & 0 \\ 0 & 3 \\ 0 & 0 \end{pmatrix} \quad \mathbf{D}_{w1} = \begin{pmatrix} 0 & 0 \\ 0.8 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\mathbf{D}_{w2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0.1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

One can notice that the subsystem $(\mathbf{E}, \mathbf{A}_1)$ is impulsive. The finite spectrum of the two subsystems is significantly different since we have $\sigma_f(\mathbf{E}, \mathbf{A}_1) = \{-0.080, -0.564\}$ and $\sigma_f(\mathbf{E}, \mathbf{A}_2) = \{-0.935, 0.126, 0.996\}$, thus the global system is not close to a linear system. The sampling time is $t_s = 0.03$ s. The activating functions $h_i(\mathbf{w}_k)$ are defined by $h_{1k} = (1 + \tanh(u_{1k}/10))/2$ and $h_{2k} = 1 - h_{1k}$. The disturbances w_{1k} and w_{2k} are bounded centred white noise, with norm bound $v_1 = v_2 = 1$. The fault signals represent control input dysfunctions and they are defined by

$$f_{1k} = \begin{cases} -0.8 u_{1k}, & \text{if } 35 \leq t_k \leq 40 \\ 0, & \text{else} \end{cases}$$

$$f_{2k} = \begin{cases} -0.8 u_{2k}, & \text{if } 40 \leq t_k \leq 45 \\ 0, & \text{else} \end{cases}$$

The first UIADO is designed with the first control input as UI, the \mathcal{L}_2 -gain from \mathbf{w} to the first and second output estimation error are $g_{11} = 0.098$ and $g_{12} = 0.050$, respectively. The second UIADO is designed with the second disturbance as UI, the \mathcal{L}_2 -gain from w_1 to the first and second output estimation error are $g_{21} = 0.052$ and $g_{22} = 0.009$, respectively.

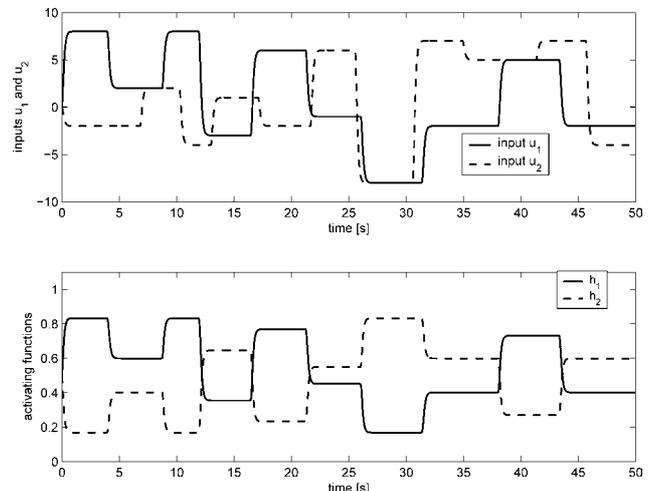


Fig. 1 Inputs and activating functions of the simulated system

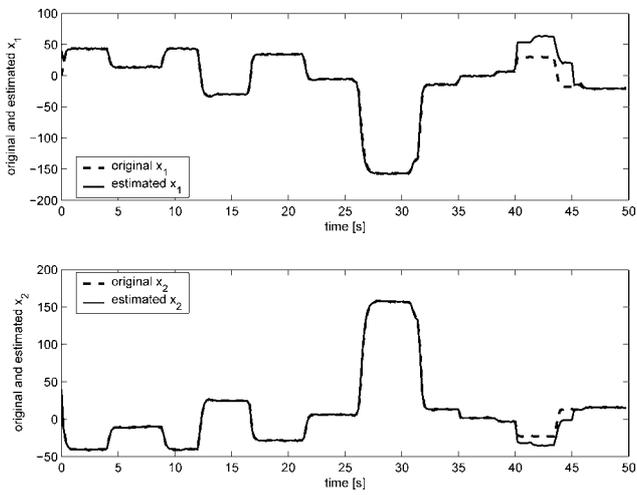


Fig. 2 Original and estimated x_{1k} and x_{2k} obtained with an observer decoupling the first input

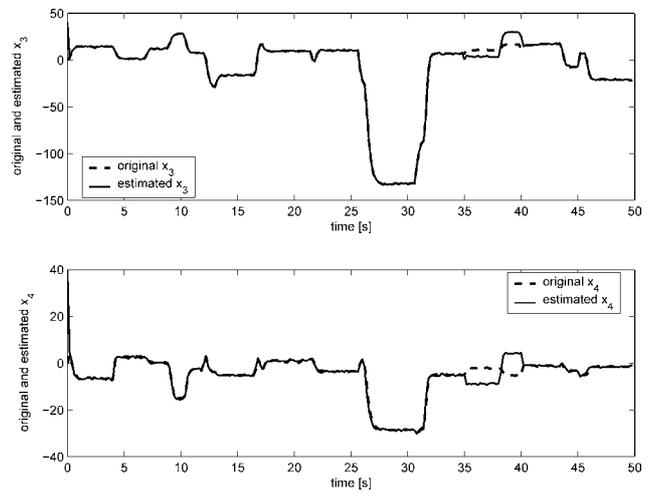


Fig. 5 Original and estimated x_{3k} and x_{4k} obtained with an observer decoupling the second input

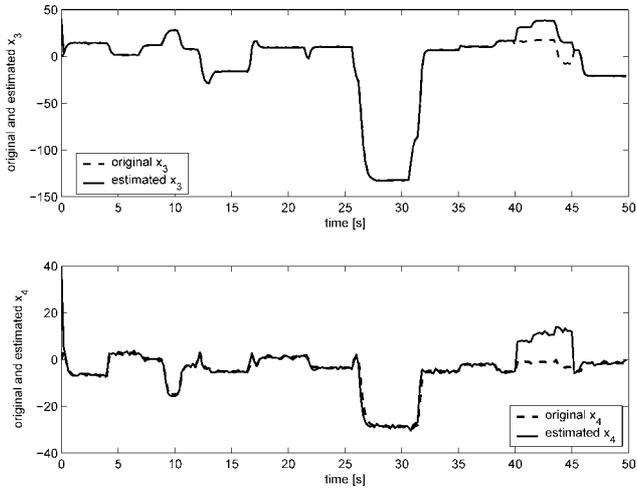


Fig. 3 Original and estimated x_{3k} and x_{4k} obtained with an observer decoupling the first input

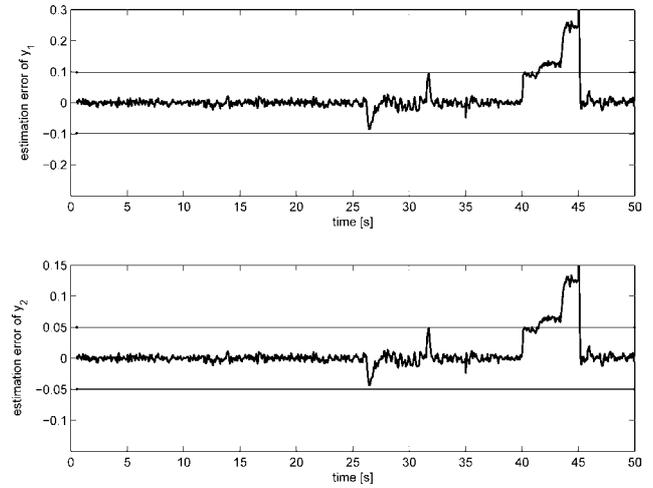


Fig. 6 Output estimation errors obtained with an observer decoupling the first input

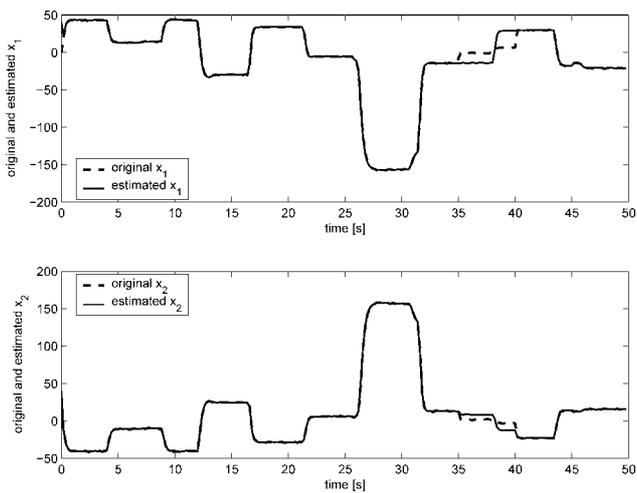


Fig. 4 Original and estimated x_{1k} and x_{2k} obtained with an observer decoupling the second input

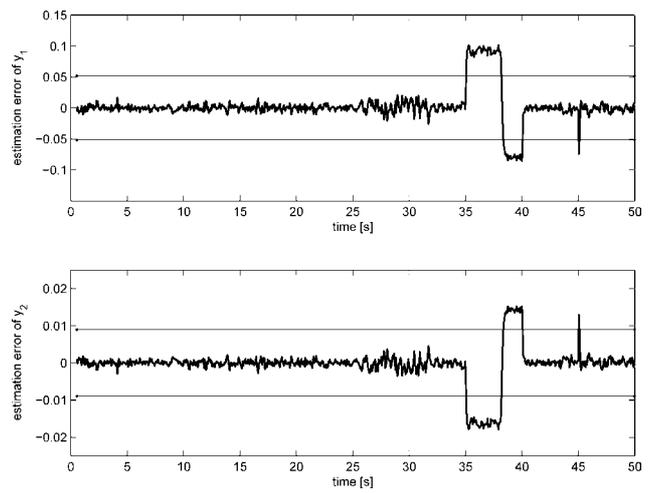


Fig. 7 Output estimation errors obtained with an observer decoupling the second input

In Fig. 1, the inputs and the activating functions are displayed. Figs. 2 and 3 display the comparison of the state variables and their estimates supplied by the UIDAO insensitive to the first fault. The fault f_{1k} appearing between $t_k = 35$ s and $t_k = 40$ s does not affect the

estimation, whereas the estimation is sensitive to the fault f_{2k} present between $t_k = 40$ s and $t_k = 45$ s. Figs. 4 and 5 display the comparison of the state variables and their estimates supplied by the UIDAO insensitive to the fault f_{2k} and affected by f_{1k} . The residual signals and

their corresponding threshold, for $\alpha = 1$, are displayed in Figs. 6 and 7.

The residual signals computed with the output estimation error of the first observer are sensitive to f_{2k} and insensitive to f_{1k} whereas the residual signals computed with second UIADO are sensitive to f_1 and insensitive to f_2 . The \mathcal{L}_2 -gains g_{11} , g_{12} , g_{21} and g_{22} are good thresholds for fault isolation, with $\alpha = 1$, since the fault f_{2k} (occurring for $40 \leq t_k \leq 45$) is isolated at $t = 41.5$ s, and the fault f_{1k} (occurring for $35 \leq t_k \leq 40$) is isolated at $t = 35.1$ s. The sudden appearance or disappearance of a fault may cause abrupt changes of the state variables that the estimate cannot follow instantaneously. Thus, even if the estimation is decoupled from the occurring fault, a residual signal may transiently be higher than the threshold. This phenomenon appears in Fig. 7 at $t = 45$ when f_{2k} disappears and causes a brief overshoot of the output estimation error.

8 Conclusion

In this paper, a simple method is proposed to design UIO for TSDS. Sufficient existence conditions were given, and the determination of the observer parameters is based on solving a system of strict LMI. If the unknown input decoupling condition is not satisfied, it has been proposed to design an \mathcal{L}_2 -observer in order to minimise the \mathcal{L}_2 -gain from the UI to the estimated state. A compromise between perfect unknown input decoupling and unknown input attenuation can be made to design observer ensuring perfect decoupling face to a subset of the unknown inputs, and robustness face to the other unknown inputs. The three observer designs are treated in both continuous and discrete-time cases. The proposed observers are used to perform fault diagnosis. Designing a bank of observers, where each observer considers a fault as a UI, the GOS can be extended to TSDS.

9 Reference

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