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Design of Proportional-Integral Observer for Unknown Input Descriptor Systems

Damien Koenig and Saïd Mammar

***Abstract*—This note presents simple methods to design full- and reduced-order proportional integral observer for unknown inputs (UI) descriptor systems. Sufficient conditions for the existence of the observer are given and proven. The observer is solvable by any pole placement algorithm, it achieves *a posteriori* robustness state and UI estimation versus to time varying parameters and bounded nonlinear UI. An illustrative example is included.**

***Index Terms*—Descriptor system, nonlinearities and robustness, state and unknown inputs (UI) estimation.**

I. Introduction

Since Luenberger's work [9] the problem of designing observers for linear state–space models has been dealt with intensively. There are many approaches designing an observer for linear time invariant descriptor systems such as Luenberger observers [7] or observers in descriptor form [4]. In [14], the method, based on the singular value decomposition and the concept of generalized inverse matrix, has been proposed to design a reduced-order observer. In [15], the generalized Sylvester equation was used to develop a procedure designing reduced-order observers. In [12], a method based on the generalized inverse matrix, which extends the method developed in [5], has been described. Full- and reduced-order observers for discrete-time descriptor systems have been presented in [3] and [4]. In [2], the method designs full and reduced order observers for unknown inputs (UI) free linear time-invariant (LTI) descriptor systems. The full-order observer approach is based on a method developed in [1], it extends to descriptor systems, while the reduced-order observer design method leans on the resolution of the generalized Sylvester equation.

In this note, we present simple and new methods to design full and reduced order proportional integral (PI) observers for UI descriptor systems subject to parameter variations. The full-order observer approach is based on a method developed in [2], it is extended to PI observer for a UI descriptor system. The PI observer structure sticks to the structure proposed in [13]. Meanwhile, the reduced order observer approach is performed through a coordinate system transformation and some substitutions, which were treated in [10] and extended to a PI observer for UI descriptor system.

We consider a class of UI LTI descriptor systems described by

$$\begin{cases} E^*\dot{x} = A^*x + B^*u + N^*f \\ y^* = C^*x \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the known input vector, $y^* \in \mathbb{R}^m$ is the output vector, $f \in \mathbb{R}^{n_f}$ is the disturbance (or UI) vector with its distribution matrix $N^* \in \mathbb{R}^{e\times n_f}$. $E^* \in \mathbb{R}^{e\times n}$, $A^* \in \mathbb{R}^{e\times n}$, $B^* \in \mathbb{R}^{e\times n_u}$, and $C^* \in \mathbb{R}^{m\times n}$ are known constant matrices. Let $r := \operatorname{rank} E^* \le \min\{e, n\}$, and assume that $\operatorname{rank} N^* = n_f$ and $\operatorname{rank} C^* = m$. As in [8], the regularity assumption [i.e., A^*, E^* are square and $\det(\lambda E^* - A^*) \neq 0$] is not required. Moreover if $E^* = I$, then (8) is always verified, although the UI decoupled condition [i.e., $\operatorname{rank}(C^*N^*) = \operatorname{rank} N^* = n_f$] needed in UI observer [1], [6] might not be. Therefore, the PI observer presents generally less restrictive existence conditions than the UI observer [1], [6].

The note is organized as follows. Section II presents the general problem statement and assumptions. Section III is dedicated to the design procedure and existence conditions for the PI-observer, which are established and proven. Finally, Section IV applies the developed procedure to a design example that includes parameter variations and UI nonlinear bounded function. The results obtained are compared to those of the P-observer of [2].

II. Problem Statement and Assumptions

Some useful definitions concerning observability of singular systems are reminded

$$\begin{cases} E\dot{x} = Ax \\ y = Cx. \end{cases} \tag{2}$$

If an arbitrary initial condition is permitted, the free response of (2) may contain impulsive modes. If $\deg[\det(sE - A)] = q$, the free-response of $\{E\dot{x} = Ax\}$ exhibits exponential modes at q finite frequencies and $\operatorname{rank}(E) - q$ impulsive modes which are undesirable. The following definitions are taken from [4].

Definition 1: System (2) is called R-detectable if there exists a matrix L such that $\sigma(E, A - LC) \subset \mathbb{C}^-$, where $\sigma(E, A - LC) := \{s\colon s \in \mathbb{C},\ s \text{ finite}, \det(sE - (A - LC)) = 0\}$, $\subset :=$ {be included

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D. Koenig is with the Laboratoire d'Automatique de Grenoble 38402 Saint Martin d'Hères, Cedex, France (e-mail: Damien.Koenig@esisar.inpg.fr).

S. Mammar is with CEMIF-LSC, FRE-CNRS 2494, Laboratoire de Systèmes Complexes, Université d'Évry Val-d'Essonne, 91020, Evry, Cedex, France (e-mail: smam@iup.univ-evry.fr).

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in} and $\mathbb{C}^- := \{$the open left-half complex plane$\}$. From [2] and [4], singular system (2) is R-observable (resp., R-detectable) if and only if $\operatorname{rank} [\, sE^T - A^T \quad C^T \,]^T = n$, for all $s \in \mathbb{C}$ [resp., $\operatorname{Re}(s) \geq 0$].

Definition 2: A pair of matrices (E, A) is said to be stable and impulse-free (causal), if $\sigma(E, A) \subset \mathbb{C}^-$ and $\operatorname{rank} E = \deg(\det(sE - A))$, respectively.

Definition 3: System (2) is called impulse observable if there exists a matrix L such that $\operatorname{rank}(E) = \deg(\det(sE - (A - LC)))$ or, equivalently, if

$$\operatorname{rank} \begin{bmatrix} E & A \\ 0 & E \\ 0 & C \end{bmatrix} = n + \operatorname{rank} E.$$

Definition 4: System (2) is observable if it is both R-observable and $\operatorname{rank} [\, E^T \quad C^T \,]^T = n$. A system is impulse observable if it is observable, but its inverse is not true. $\operatorname{rank} [\, E^T \quad C^T \,]^T = n$ implies that the impulse observability is true.

Since the observer is a PI observer we can approximate the bounded nonlinear UI function (f) by a step function where the approximate error can be minimized by increasing the observer bandwidth (see remark 2). The PI observer (5) is also synthesized on the basis of the following augmented descriptor system:

$$\begin{cases} \begin{bmatrix} E^* & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{f} \end{bmatrix} = \begin{bmatrix} A^* & N^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} + \begin{bmatrix} B^* \\ 0 \end{bmatrix} u \\ y^* = [\, C^* \quad 0 \,] \begin{bmatrix} x \\ f \end{bmatrix}. \end{cases} \tag{3}$$

Assuming that $\operatorname{rank} [\, E^* \quad N^* \,] = \operatorname{rank} E^* = r$, there exists a nonsingular matrix P^* such as: $P^* [\, E^* \quad N^* \,] = \begin{bmatrix} E & N \\ 0 & 0 \end{bmatrix}$; $P^* A^* = \begin{bmatrix} A \\ A_1 \end{bmatrix}$; $P^* B^* = \begin{bmatrix} B \\ B_1 \end{bmatrix}$; $E \in \mathbb{R}^{r \times n}$, and $\operatorname{rank} E = r$.

Using the aforementioned transformations, (3) is reduced to a restricted systems equivalence (r.s.e.)

$$\begin{cases} \begin{bmatrix} E & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{f} \end{bmatrix} = \begin{bmatrix} A & N \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ f \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \\ y = [\, C \quad 0 \,] \begin{bmatrix} x \\ f \end{bmatrix} \end{cases} \tag{4}$$

where $y = \begin{bmatrix} -B_1 u \\ y^* \end{bmatrix} \in \mathbb{R}^{q=m+e-r}$ and $C = \begin{bmatrix} A_1 \\ C^* \end{bmatrix} \in \mathbb{R}^{q \times n}$. If $\operatorname{rank} E^* = e$ or $\operatorname{rank} \begin{bmatrix} E^* \\ C^* \end{bmatrix} = n$ then $E = E^*$, $N = N^*$, $A = A^*$, $B = B^*$, $C = C^*$, and $y = y^*$. Assuming that (4) is impulse observable and R-detectable, the following observer can be designed:

$$\begin{cases} \dot{z} = Fz + L_1 y + L_2 y + Ju + T_1 N \hat{f} \\ \dot{\hat{f}} = L_3 (y - \hat{y}) \\ \hat{x} = M_1 z + T_2 y \\ \hat{y} = C \hat{x}. \end{cases} \tag{5}$$

The second equation in (5) describes the integral loop added to the proportional one, in the first equation. This observer type is therefore termed PI observer. Matrices F, L_1, L_2, L_3, J, M_1, T_1, and T_2 are determined in such a way to enable the asymptotical convergence to zero of the state and UI estimation errors, respectively defined by $e = x - \hat{x}$ and $e_f = f - \hat{f}$.

Remark 1: If $\operatorname{rank} [\, E^* \quad N^* \,] = \operatorname{rank} E^* = r$ then there exists a regular matrix P^* such that the impulse observability condition of system (3) is equivalent to the impulse observability condition of the rse (4) and such that the matrix E is full-row rank (i.e., $\operatorname{rank} E = r$). Moreover, if (4) is impulse observable, then it is easy to verify, since E is full-row rank, that $\operatorname{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$. There exists then a full-row rank matrix $[\, T_1 \quad T_2 \,] = \begin{bmatrix} E \\ C \end{bmatrix}^{+}$[1] such that

$$[\, T_1 \quad T_2 \,] \begin{bmatrix} E \\ C \end{bmatrix} = I_n. \tag{6}$$

On the other hand, if $\operatorname{rank} [\, E^T \quad C^T \,]^T = n$, then (4) is impulse observable (cf. Definition 4).

III. Design Procedure

A. Full-Order Observer Design

We want to design the full-order (i.e., $n + n_f$) PI observer (5) with $M_1 = I_n$. Based on the proportional observer designed by [2] and the PI observer [13] designed for nonsingular systems, we can propose the full-order nonsingular PI observer (5). Combining (4) to (6) and using the equality $e = (I_n - T_2 C)x - z = T_1 E x - z$, we obtain the time derivative of the state and UI estimation errors

$$\begin{bmatrix} \dot{e} \\ \dot{e}_f \end{bmatrix} = \begin{bmatrix} F & T_1 N \\ -L_3 C & 0 \end{bmatrix} \begin{bmatrix} e \\ e_f \end{bmatrix} + \begin{bmatrix} T_1 A - F T_1 E - L_1 C - L_2 C \\ 0 \end{bmatrix} x + \begin{bmatrix} T_1 B - J \\ 0 \end{bmatrix} u \tag{7}$$

where $F = T_1 A - L_2 C$. Letting $L_1 = F T_2$, $J = T_1 B$, an autonomous system is obtained. Estimation errors converge asymptotically to zero if matrix $A_{obs} = \begin{bmatrix} T_1 A - L_2 C & T_1 N \\ -L_3 C & 0 \end{bmatrix}$ is Hurwitz. This matrix becomes $A_{obs} = \begin{bmatrix} T_1 A & T_1 N \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L_2 \\ L_3 \end{bmatrix} [\, C \quad 0 \,]$, and it can be stabilized by the gain $\begin{bmatrix} L_2 \\ L_3 \end{bmatrix}$, if and only if the pair $\left(\begin{bmatrix} T_1 A & T_1 N \\ 0 & 0 \end{bmatrix}, [\, C \quad 0 \,] \right)$ is detectable, i.e.,

$$\operatorname{rank} \begin{bmatrix} sI_n - T_1 A & -T_1 N \\ 0 & sI_{n_f} \\ C & 0 \end{bmatrix} = n + n_f$$

for all $s \in \mathbb{C}$, $\operatorname{Re}(s) \geq 0$.

The following theorems summarize the previous derivations and give the sufficient conditions for the existence of the full-order PI observer (5) for s (3) and (4), respectively.

Theorem 1: The sufficient conditions for the existence of the PI observer (5) for (3) are

$$\begin{cases} \operatorname{rank} [\, E^* \quad N^* \,] = \operatorname{rank} E^* = r \\ \text{or } \operatorname{rank} E^* = e \\ \text{or } \operatorname{rank} \begin{bmatrix} E^* \\ C^* \end{bmatrix} = n \end{cases} \tag{8}$$

$$\operatorname{rank} \begin{bmatrix} E^* & A^* \\ 0 & E^* \\ 0 & C^* \end{bmatrix} = n + \operatorname{rank} E^* \tag{9}$$

$$\operatorname{rank} \begin{bmatrix} sE^* - A^* & -N^* \\ 0 & sI_{n_f} \\ C^* & 0 \end{bmatrix} = n + n_f \quad \text{for all } s \in \mathbb{C}, \qquad \operatorname{Re}(s) \geq 0. \tag{10}$$

Condition (8) ensures that it exists a regular matrix P^* which transforms (3) into (4). Equation (9) is the impulse observability condition and (10) is the generalization of R-detectability condition for singular systems, which is derived from [11] for nonsingular systems (i.e., $E^* = I$).

[1]Notation: $(.)^+$ is the generalized inverse of $(.)$ verifying the Moore–Penrose conditions $(.)(.)^+(.) = (.)$ and $(.)^+(.)(.)^+ = (.)^+$.

Theorem 2: The sufficient conditions for the existence of the PI observer (5) for (4) are

$$\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} = n \tag{11}$$

$$\operatorname{rank}\begin{bmatrix} sE - A & -N \\ 0 & sI_{n_f} \\ C & 0 \end{bmatrix} = n + n_f$$
$$\text{for all } s \in \mathbb{C}, \qquad \operatorname{Re}(s) \geq 0. \tag{12}$$

The condition that A_{obs} can be stabilized is given by the following theorem.

Theorem 3: System (7) is stabilizable if and only if

$$\operatorname{rank}\begin{bmatrix} sI_n - T_1 A & -T_1 N \\ 0 & sI_{n_f} \\ C & 0 \end{bmatrix} = n + n_f$$
$$\text{for all } s \in \mathbb{C}, \qquad \operatorname{Re}(s) \geq 0. \tag{13}$$

Proof: Under (8) and (9), this condition is equivalent to (10) and (12). This is proven in the Appendix. From (1), the other parts are obvious. ■

Remark 2: If the UI is not a step function but a bounded nonlinear UI, it is advisable to always set $\dot{f} = 0$ and increase the bandwidth of the observer; this compensates the approximation error model of the UI (see Figs. 7 and 8 for illustration). Whereas the increase of the bandwidth implies a more noisy state estimation error. A compromise between robustness and sensitivity should be treated *a posteriori*.

Algorithm 1: The problem of designing a full-order PI observer for an LTI system affected by unmeasured disturbances can be formulated as a simple pole placement algorithm. Let $\dot{f} = 0$ and transform system (1) into (4). Let $[T_1 \quad T_2] = [\begin{smallmatrix} E \\ C \end{smallmatrix}]^+$ and find matrices $[\begin{smallmatrix} L_2 \\ L_3 \end{smallmatrix}]$ such that matrix $([\begin{smallmatrix} T_1 A & T_1 N \\ 0 & 0 \end{smallmatrix}] - [\begin{smallmatrix} L_2 \\ L_3 \end{smallmatrix}][C \quad 0])$ is stable. Finally compute the matrices $F = T_1 A - L_2 C$, $L_1 = F T_2$, $J = T_1 B$ and $M_1 = I_n$.

B. Reduced Order Observer Design

Based on the results of [10], we now propose to design the reduced-order (i.e., $r + n_f$) PI observer (5). Since E is a full-row rank, there exists a regular matrix $P = [E^+ \quad P_1]$, which transforms E into the following form:

$$E[E^+ \quad P_1] = [I_r \quad 0_{r\times(n-r)}] \tag{14}$$

where $P_1 = \operatorname{Ker}(E) \in \mathbb{R}^{n\times(n-r)}$, $E^+ = E^T(EE^T)^{-1}$, $P_1^T P_1 = I_{n-r}$, and $P^{-1} = [E^T \quad P_1]^T$.

As the matrix P is regular, then (4) is equivalent to the following subsystem:

$$\begin{cases} \dot{\overline{x}}_1 = \overline{A}_1 \overline{x}_1 + \overline{A}_2 \overline{x}_2 + Bu + Nf \\ \dot{f} = 0 \\ y = \overline{C}_1 \overline{x}_1 + \overline{C}_2 \overline{x}_2 \end{cases} \tag{15}$$

where $[\begin{smallmatrix} \overline{x}_1 \\ \overline{x}_2 \end{smallmatrix}] = P^{-1}x$, $\overline{x}_1 \in \mathbb{R}^r$, $[\overline{A}_1 \quad \overline{A}_2] = AP$ and $[\overline{C}_1 \quad \overline{C}_2] = CP$.

In addition, the matrix $\overline{C}_2 \in \mathbb{R}^{q\times(n-r)}$ is full-column rank, if and only if system (4) is input observable. Assuming $\operatorname{rank}(\overline{C}_2) = n - r$, we can, therefore, make the regular $P_2 = [\begin{smallmatrix} P_3 \\ \overline{C}_2^+ \end{smallmatrix}]$ transformation

$$P_2 \overline{C}_2 = \begin{bmatrix} P_3 \\ \overline{C}_2^+ \end{bmatrix} \overline{C}_2 = \begin{bmatrix} 0_{(q-(n-r))\times(n-r)} \\ I_{n-r} \end{bmatrix} \tag{16}$$

where $P_3^T = \operatorname{Ker}(\overline{C}_2^T) \in \mathbb{R}^{q\times(q-(n-r))}$ and $\overline{C}_2^+ = (\overline{C}_2^T \overline{C}_2)^{-1} \overline{C}_2^T$.

Multiplying both sides of the measurement equation of (15) by P_2, we get

$$P_3 y = P_3 \overline{C}_1 \overline{x}_1$$
$$\overline{x}_2 = \overline{C}_2^+ y - \overline{C}_2^+ \overline{C}_1 \overline{x}_1 \tag{17}$$

hence, substituting (17) into (15), we obtain the following subsystem:

$$\begin{cases} \dot{\overline{x}}_1 = \left(\overline{A}_1 - \overline{A}_2 \overline{C}_2^+ \overline{C}_1\right) \overline{x}_1 + \overline{A}_2 \overline{C}_2^+ y + Bu + Nf \\ \dot{f} = 0. \end{cases} \tag{18}$$

We can now propose the following natural observer structure:

$$\begin{cases} \dot{\hat{\overline{x}}}_1 = \left(\overline{A}_1 - \overline{A}_2 \overline{C}_2^+ \overline{C}_1\right) \hat{\overline{x}}_1 + \overline{A}_2 \overline{C}_2^+ y + Bu \\ \qquad + N\hat{f} + \overline{L}_2 \left(P_3 y - P_3 \overline{C}_1 \hat{\overline{x}}_1\right) \\ \dot{\hat{f}} = \overline{L}_3 P_3 (y - \hat{y}) = \overline{L}_3 P_3 \overline{C}_1 (\overline{x}_1 - \hat{\overline{x}}_1) \\ \hat{\overline{x}}_2 = \overline{C}_2^+ y - \overline{C}_2^+ \overline{C}_1 \hat{\overline{x}}_1 \\ \hat{x} = E_1^+ \hat{\overline{x}}_1 + P_1 \hat{\overline{x}}_2 = \left(E^+ - P_1 \overline{C}_2^+ \overline{C}_1\right) \hat{\overline{x}}_1 + P_1 \overline{C}_2^+ y \end{cases} \tag{19}$$

which is equivalent to the reduced-order PI observer (5) where $z = \hat{\overline{x}}_1$, $F = (\overline{A}_1 - \overline{A}_2 \overline{C}_2^+ \overline{C}_1 - \overline{L}_2 P_3 \overline{C}_1)$, $L_1 = \overline{A}_2 \overline{C}_2^+$, $L_2 = \overline{L}_2 P_3$, $L_3 = \overline{L}_3 P_3$, $J = B$, $M_1 = (E^+ - P_1 \overline{C}_2^+ \overline{C}_1)$, $T_2 = P_1 \overline{C}_2^+$, and $T_1 = I_r$.

Let $\overline{e}_1 = \overline{x}_1 - \hat{\overline{x}}_1$ and combine (17)–(19) to obtain the autonomous state estimation errors system

$$\begin{cases} \begin{bmatrix} \dot{\overline{e}}_1 \\ \dot{e}_f \end{bmatrix} = \left(\begin{pmatrix} \left(\overline{A}_1 - \overline{A}_2 \overline{C}_2^+ \overline{C}_1\right) & N \\ 0 & 0 \end{pmatrix} - \begin{bmatrix} \overline{L}_2 \\ \overline{L}_3 \end{bmatrix} [P_3 \overline{C}_1 \quad 0] \right) \begin{bmatrix} \overline{e}_1 \\ e_f \end{bmatrix} \\ \begin{bmatrix} e \\ e_f \end{bmatrix} = \begin{bmatrix} E^+ - P_1^+ \overline{C}_2^+ \overline{C}_1 & 0 \\ 0 & I_{n_f} \end{bmatrix} \begin{bmatrix} \overline{e}_1 \\ e_f \end{bmatrix}. \end{cases} \tag{20}$$

The autonomous system (20) can be stabilized by the matrix gain $[\begin{smallmatrix} \overline{L}_2 \\ \overline{L}_3 \end{smallmatrix}]$ if and only if the pair $([\begin{smallmatrix} (\overline{A}_1 - \overline{A}_2 \overline{C}_2^+ \overline{C}_1) & N \\ 0 & 0 \end{smallmatrix}], [P_3 \overline{C}_1 \quad 0])$ is detectable.

The following theorems summarize the aforementioned derivations and give the sufficient conditions for the existence of the reduced order PI observer (5) for (18).

Theorem 4: The sufficient conditions for the existence of the reduced-order PI observer (5) for (4) are

$$(12) \quad \text{and} \quad \operatorname{rank} E = r, \qquad E \in \mathbb{R}^{r\times n}. \tag{21}$$

Theorem 5: The sufficient conditions for the existence of the PI observer (5) for (18) are

$$\operatorname{rank} \overline{C}_2 = n - r \tag{22}$$

$$\operatorname{rank}\begin{bmatrix} sI_n - \left(\overline{A}_1 - \overline{A}_2 \overline{C}_2^+ \overline{C}_1\right) & -N \\ 0 & sI_{n_f} \\ P_3 \overline{C}_1 & 0 \end{bmatrix} = r + n_f$$
$$\text{for all } s \in \mathbb{C}, \qquad \operatorname{Re}(s) \geq 0. \tag{23}$$

Proof: See the Appendix for the proof of the equivalences between (11) and (22) and under $\operatorname{rank} E = r$ and (11), the equivalence between (12) and (23). ■

Algorithm 2: Let $\dot{f} = 0$, transform (1) into (4). Let $P = [E^+ \quad \ker E]$, $[\overline{A}_1 \quad \overline{A}_2] = AP$, $[\overline{C}_1 \quad \overline{C}_2] = CP$, $P_3^T = \ker \overline{C}_2^T$ and find matrices $[\begin{smallmatrix} \overline{L}_2 \\ \overline{L}_3 \end{smallmatrix}]$ such that matrix $([\begin{smallmatrix} (\overline{A}_1 - \overline{A}_2 \overline{C}_2^+ \overline{C}_1) & N \\ 0 & 0 \end{smallmatrix}] - [\begin{smallmatrix} \overline{L}_2 \\ \overline{L}_3 \end{smallmatrix}][P_3 \overline{C}_1 \quad 0])$ is stable. Remaining matrices are obtained as $F = (\overline{A}_1 - \overline{A}_2 \overline{C}_2^+ \overline{C}_1 - \overline{L}_2 P_3 \overline{C}_1)$, $L_1 = \overline{A}_2 \overline{C}_2^+$, $L_2 = \overline{L}_2 P_3$, $L_3 = \overline{L}_3 P_3$, $J = B$, $M_1 = (E^+ - P_1 \overline{C}_2^+ \overline{C}_1)$, $T_2 = P_1 \overline{C}_2^+$ and $T_1 = I_r$.

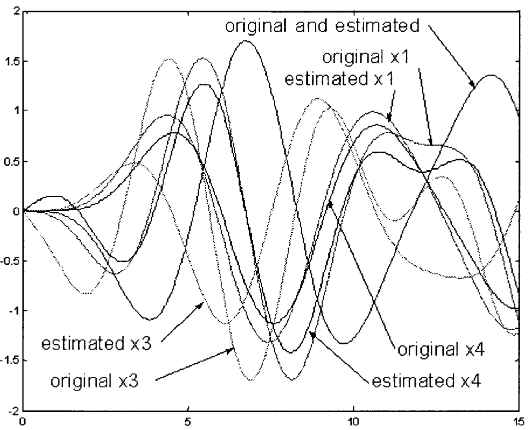

Fig. 1. State estimation with P-observer.

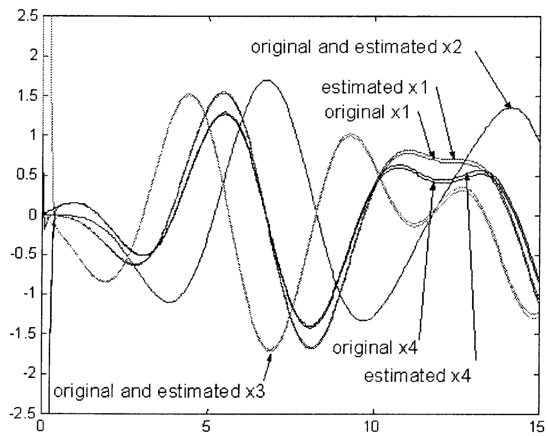

Fig. 2. State estimation with PI-observer.

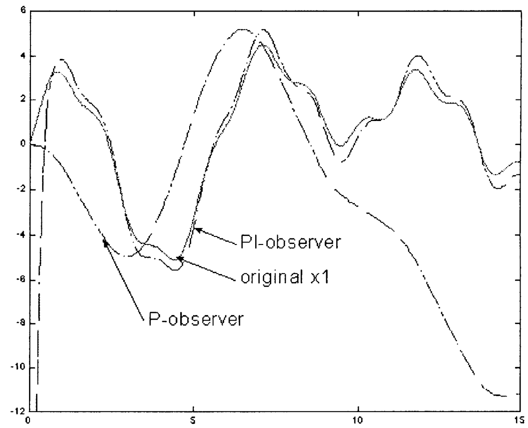

Fig. 3. Estimates for x_1.

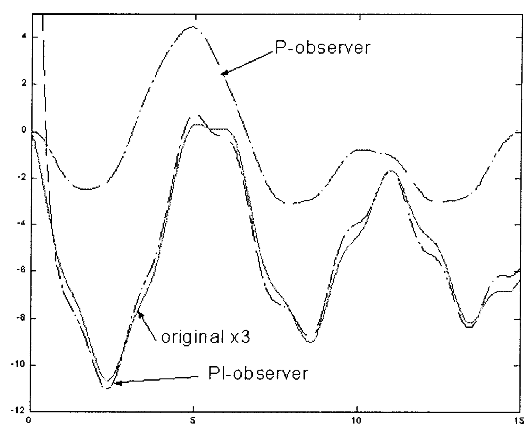

Fig. 4. Estimates for x_3.

IV. Example

In this section, two cases are being examined. In the first one, a linear system is subject to parameter variations. It illustrates the estimation performance of PI observer and compares it to the P observer developed in [2]. In the second one, the system is also disturbed by nonlinear UI. The results show that the PI observer estimates both the state and the bounded nonlinear UI under parameter variations. Some technical issues for the choice of the eigenvalues of the PI-observer are discussed, but due to space limitation only the full-order observer is illustrated.

Consider the UI-LTI descriptor system (1) described by

$$E^* = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \quad A^* = \begin{bmatrix}0&0&1&0\\1&0&0&0\\-1&0&0&1\\0&1&1&1\end{bmatrix}$$

$$B^* = \begin{bmatrix}0\\0\\0\\-1\end{bmatrix} \quad N^* = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \quad C^* = [0 \quad 1 \quad 0 \quad 0].$$

A. Case 1. Uncertain Parameters

In this case, we consider the system affected by an uncertain additive matrix given by

$$\tilde{A}^* = A^*(I + \Delta A)$$

where $\tilde{a}_{ij} = a_{ij}(1 + \Delta a_{ij})$, $\Delta a_{ij} = \delta_{ij}\sin(2\pi\delta'_{ij}t)$ and $|\delta'_{ij}| < 2$, $|\delta_{ij}| < 10$.

$\mathrm{Rank}\begin{bmatrix}E^*\\C^*\end{bmatrix} = 2 \neq n$, but $\mathrm{rank}\,[E^* \quad N^*] = \mathrm{rank}\,E^*$, then (8) holds and (4) exists. Since the pair $\left(\begin{bmatrix}T_1A & T_1N\\0 & 0\end{bmatrix}, [C \quad 0]\right)$ is observable, it is possible to assign all eigenvalues to an arbitrary set values. In the following two figures, we compare the results obtained with the P-observer and PI-observer. In these cases, the eigenvalues chosen for the two observers are the same and set at $\Lambda = \{-22, -18, -26, -19, -33\}$. As shown on Figs. 1 and 2, the PI-observer leads to a better estimation than the P-observer. It is also demonstrated that our PI observer is robust against to parameter variations.

B. Case 2. Nonlinear Disturbance and Uncertain Parameters

We compare the results obtained with the PI-observer and the P-observer, both when the parameters of $\tilde{A}^*$ are uncertain and time varying (same as in the first case), and also when the system is disturbed by a bounded nonlinear UI. The results obtained with the P-observer and the PI-observer are given on Figs. 3–6. These figures indicate that the PI-observer performs a much better estimate than the P-observer. Moreover, the PI-observer provides a good and robust estimation of the UI ($f = 5 + 2\sin(4t)$), which the P-observer cannot do.

This example shows that the PI observer proposed, gives *a posteriori* state and nonlinear bounded UI robust estimations versus of parameter variation. On Figs. 3–6, the eigenvalues of the observer are fixed at $\Lambda = \{-22, -18, -26, -19, -33\}$ and the results can be significantly improved by setting higher negative eigenvalues (Remark 2, Fig. 7). This causes unbiased nonlinearity estimation, but it obviously increases the

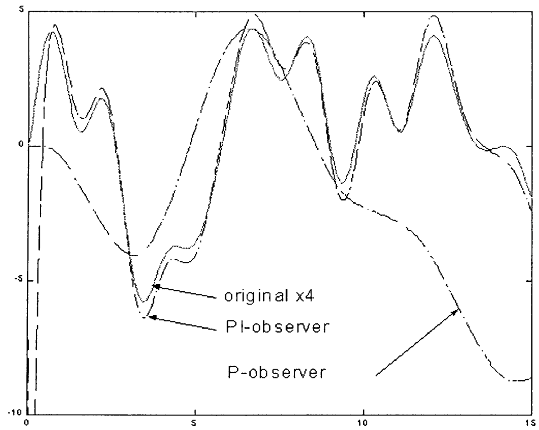

Fig. 5. Estimates for x_4.

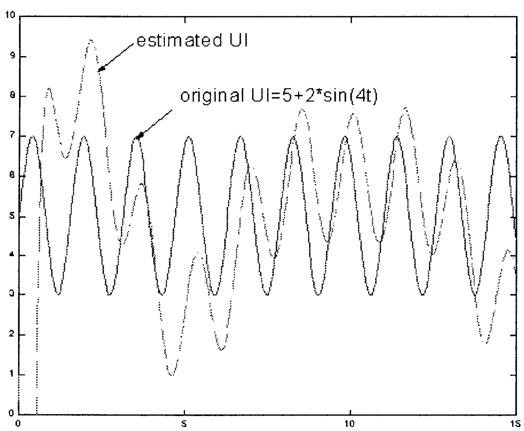

Fig. 6. Estimate for UI.

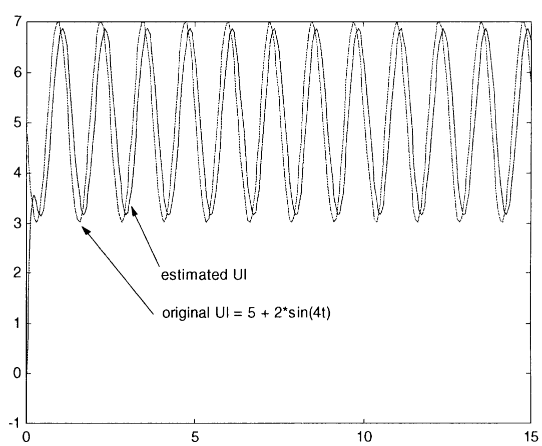

Fig. 7. Estimate for UI, with 2 Λ.

sensitivity to noises due to an increased bandwidth. Finally, Fig. 8 shows that, increasing the bandwidth (i.e., resp., $1\ \Lambda,\ 10\ \Lambda,\ 100\ \Lambda$) of the observer implies a reduction of the phase lag and an attenuation of the transfer function between $\hat{f}\ (e_f)$ and $\hat{x}_1\ (e_1)$ at low frequencies where

$$\hat{x}_1 = (1 \quad 0 \quad 0 \quad 0)(sI - F)^{-1} T_1 N \hat{f}\Big|_{y=u=0}$$
$$e_1 = (1 \quad 0 \quad 0 \quad 0)(sI - F)^{-1} T_1 N e_f.$$

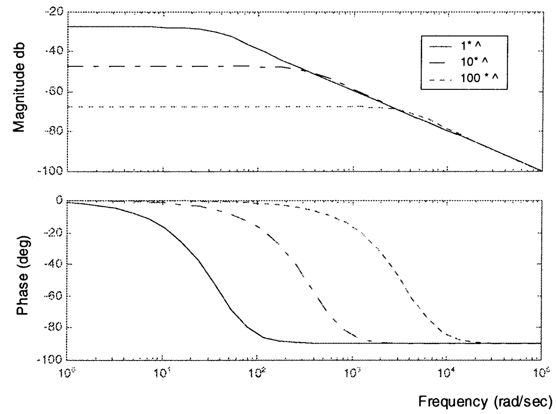

Fig. 8. Bode between UI $\hat{f}$ and estimate $\hat{x}_1$, with Λ.

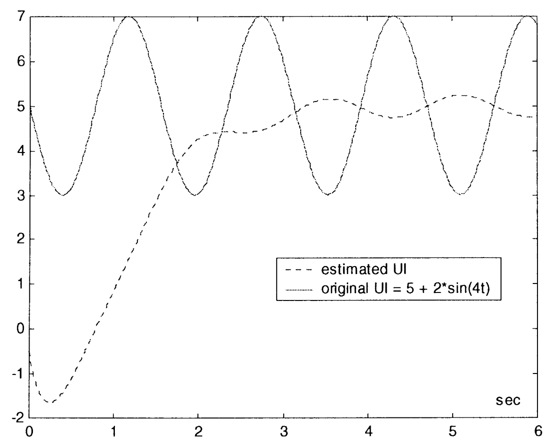

Fig. 9. Estimate for UI, with $10^{-1}\ \Lambda$.

The same results are obtained for the other states (states estimations errors), the estimation is, thus, robust face of UI. Notice that, if the UI is not belonging to the bandwidth of the synthesized PI observer, the state and UI estimation is namely not biased on mean value. Fig. 9 illustrates this fact, the frequency of the UI is of 4 rad/s while the bandwidth of the observer obtained for 0.1 Λ is less than 3.3 rad/s.

V. Conclusion

The design of a full- and reduced-order nonsingular PI observer for UI descriptor systems and existence conditions have been given and proven. The proposed observer aims at estimating both the state and the UI. It has been shown that the existence conditions of the PI-Observer proposed, generalize those adopted in [2] for the proportional observer design of free UI descriptor system. Moreover, *a posteriori* robustness state and UI estimations face to parameter variations and UI bounded nonlinearities may be addressed by setting higher eigenvalues. This cannot be achieved with proportional observer [2]. One of the perspectives that could be worth being explored, is the use of both state and UI estimations to design an UI tolerant control system.

Appendix

Proof: Under (8) and (9), we prove that (10) $\Leftrightarrow$ (12) $\Leftrightarrow$ (13). For all $s \in \mathbb{C}$, $\mathrm{Re}(s) \geq 0$

$$\mathrm{rank} \begin{bmatrix} sE^* - A^* & -N^* \\ 0 & sI_{n_f} \\ C^* & 0 \end{bmatrix}$$

$$
\begin{aligned}
&= \operatorname{rank}\left(\begin{bmatrix} P^* & 0 & 0 \\ 0 & I_{n_f} & 0 \\ 0 & 0 & -I_q \end{bmatrix}\begin{bmatrix} sE^*-A^* & -N^* \\ 0 & sI_{n_f} \\ C^* & 0 \end{bmatrix}\right)\\
&= \operatorname{rank}\begin{bmatrix} sE-A & -N \\ 0 & sI_{n_f} \\ -C & 0 \end{bmatrix}\\
&= \operatorname{rank}\begin{bmatrix} sE-A & -N \\ 0 & sI_{n_f} \\ -sC & 0 \\ -C & 0 \end{bmatrix}\\
&= \operatorname{rank}\left(\begin{bmatrix} T_1 & 0 & T_2 & 0 \\ 0 & I_{n_f} & 0 & 0 \\ 0 & 0 & -I_q & sI_q \\ 0 & 0 & 0 & -I_q \end{bmatrix}\begin{bmatrix} sE-A & -N \\ 0 & sI_{n_f} \\ -sC & 0 \\ -C & 0 \end{bmatrix}\right)\\
&= \operatorname{rank}\begin{bmatrix} sT_1E-T_1A+sT_2C & -T_1N \\ 0 & sI_{n_f} \\ 0 & 0 \\ C & 0 \end{bmatrix}\\
&= \operatorname{rank}\begin{bmatrix} sI_n-T_1A & -T_1N \\ 0 & sI_{n_f} \\ C & 0 \end{bmatrix} = n+n_f
\end{aligned}
$$

which is equivalent to the detectability of the pair $(\begin{bmatrix} T_1A & T_1N \\ 0 & 0 \end{bmatrix}, [C \quad 0])$. ■

Proof (11) ⇔ (22): The matrix $P = [E^+ \quad P_1]$ is regular and $E[E^+ \quad P_1] = [I_r \quad 0_{r\times(n-r)}]$ then

$$
\begin{aligned}
\operatorname{rank}\begin{bmatrix} E \\ C \end{bmatrix} &= \operatorname{rank}\left(\begin{bmatrix} E \\ C \end{bmatrix}[E^+ \quad P_1]\right)\\
&= \operatorname{rank}\begin{bmatrix} I_r & 0_{r\times(n-r)} \\ \overline{C}_1 & \overline{C}_2 \end{bmatrix} = n\\
&\Leftrightarrow \operatorname{rank}\overline{C}_2 = n-r.
\end{aligned}
$$

■

Proof: Under $\operatorname{rank} E = r$ and (11), we prove that (12) ⇔ (23). Define the following regular matrices:

$$
U_1 = \begin{bmatrix} I_r & 0_{r\times q} \\ 0_{(q-(n-r))\times r} & P_3 \\ 0_{(n-r)\times r} & \overline{C}_2^+ \end{bmatrix}
$$

and

$$
V_1 = \begin{bmatrix} I_r & 0_{r\times(n-r)} \\ -\overline{C}_2^+\overline{C}_1 & I_{n-r} \end{bmatrix}
$$

$$
(12) = \begin{cases} \operatorname{rank}\left(U_1\begin{bmatrix} sE-A \\ C \end{bmatrix}PV_1\right) = n \\ \text{for all } s\in\mathbb{C},\ \operatorname{Re}(s)>0 \\ \operatorname{rank}\left(U_1\begin{bmatrix} -A & -N \\ C & 0 \end{bmatrix}\begin{bmatrix} PV_1 & 0 \\ 0 & I_{n_f} \end{bmatrix}\right) = n+n_f \\ \text{for } s=0 \end{cases}
$$

$$
= \begin{cases} \operatorname{rank}\begin{bmatrix} sI_r-\left(\overline{A}_1-\overline{A}_2\overline{C}_2^+\overline{C}_1\right) & -\overline{A}_2 \\ P_3\overline{C}_1 & 0 \\ 0 & I_{n-r} \end{bmatrix} = n \\ \text{for all } s\in\mathbb{C},\ \operatorname{Re}(s)>0 \\ \operatorname{rank}\begin{bmatrix} -\left(\overline{A}_1-\overline{A}_2\overline{C}_2^+\overline{C}_1\right) & AP_1 & -N \\ P_3\overline{C}_1 & 0 & 0 \\ 0 & I_{n-r} & 0 \end{bmatrix} = n+n_f \\ \text{for } s=0 \end{cases}
$$

$$
= \begin{cases} \operatorname{rank}\begin{bmatrix} sI_r-\left(\overline{A}_1-\overline{A}_2\overline{C}_2^+\overline{C}_1\right) \\ P_3\overline{C}_1 \end{bmatrix} = r \\ \text{for all } s\in\mathbb{C},\ \operatorname{Re}(s)>0 \\ \operatorname{rank}\begin{bmatrix} -\left(\overline{A}_1-\overline{A}_2\overline{C}_2^+\overline{C}_1\right) & -N \\ P_3\overline{C}_1 & 0 \end{bmatrix} = r+n_f \\ \text{for } s=0 \end{cases}
$$

$$
= \operatorname{rank}\begin{bmatrix} sI_n-\left(\overline{A}_1-\overline{A}_2\overline{C}_2^+\overline{C}_1\right) & -N \\ 0 & sI_{n_f} \\ P_3\overline{C}_1 & 0 \end{bmatrix} = r+n_f
$$

$$
\text{for all } s\in\mathbb{C}, \qquad \operatorname{Re}(s)\ge 0.
$$

where $P_2P_2^{-1} = \begin{bmatrix} P_3 \\ \overline{C}_2^+ \end{bmatrix}[P_3^T \quad \overline{C}_2] = \begin{bmatrix} I_{q-(n-r)} & 0 \\ 0 & I_{n-r} \end{bmatrix}$ ■

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